Vol. 41, Parts 3 and 4 December 1954

BIOMETRIKA

FOUNDED BY

W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

MANAGING EDITOR

E. S. PEARSON

ASSOCIATE EDITORS

M. G. KENDALL JOHN WISHART

in consultation with

HARALD CRAMÉR R. C. GEARY

J. B. S. HALDANE D. G. KENDALL

ISSUED BY

THE BIOMETRIKA OFFICE, UNIVERSITY COLLEGE, LONDON

PRINTED AT THE UNIVERSITY PRESS, CAMBRIDGE

[*Issued* 3 *December* 1954]

CORRIGENDA

(1) S. RUSHTON, *Biometrika* (1952), **39**, 302.

p. 305. Equation (17) should read:

$$\lambda = e^{-n\phi^2} M\left(\frac{2n-1}{2}, \frac{1}{2}, \frac{n\phi^2 t^2}{2n-2+t^2}\right).$$

p. 306. Equation (18) should read:

$$u^2 = nt^2/((2n-2)+t^2) = \frac{n^2(x_{1.}-x_{2.})^2/2}{\sum_i \sum_j (x_{ij}-x_{..})^2}.$$

(2) H. RUBEN, *Biometrika* (1954), **41**, 200.

p. 202, right-hand side of equation (16): *for* $\mathbf{L}^j \boldsymbol{\Gamma}^{m-j}$ *read* $\boldsymbol{\Gamma}^j \mathbf{B}^{m-j}$.

p. 203, 3rd line and in equation (20): *for* $\mathbf{L}^m \equiv 0$ *read* $\mathbf{L}^m \equiv \mathbf{0}$.

p. 203, right-hand side of equation (24): *for* $a_{2k+1,\,2i+1}$ *read* $a_{2k+1,\,2i-1}$.

p. 205, under $s = 4$: *for* $a_{4,0}(\alpha) = 1$ *read* $a_{4,0}(\alpha) = \dfrac{3}{\alpha^2}$.

p. 208, 3rd line from bottom: *insert* $\dfrac{3}{\alpha^2}$ *before* $\phi(0;\alpha;\beta,\gamma)$

p. 209, 2nd line, right-hand side of equation: *insert* 3 *before* $\phi(0;1;\beta,\gamma)$

p. 209, 8th line, right-hand side of equation: *for* $e^{-\frac{1}{2}u_j}$ *read* $e^{-\frac{1}{2}u_j^2}$.

p. 213, 1st line: *omit* words 'surface contents of'.

p. 215, last line but one before § 3: *for* V_{01} *read* $V_{0,1}$

p. 216, 6th line from bottom: *for* $\beta+\gamma+1$-flats *read* $\beta+\gamma-1$-flats.

p. 218, right-hand side of equation (86): *for* $(-)^{\beta+\gamma}$ *read* $(-)^{\gamma}$.

p. 219, right-hand side of equation (90): *for* $\dfrac{1}{2^{\frac{1}{2}(\beta+\gamma-1)}} \tfrac{1}{2}(\gamma-\beta)$ *read* $\dfrac{1}{2^{\beta+\gamma-1}} (-)^{\gamma} \tfrac{1}{2}(\beta-\gamma)$.

p. 219, right-hand side of equation (91): *for* $-\frac{1}{2}\ldots$ *read* $(-)^{\gamma}\frac{1}{2}\ldots.$

p. 219, right-hand side of equation (92): *for* $-(\beta-\gamma)\ldots$ *read* $(-)^{\gamma}(\beta-\gamma)\ldots.$

p. 219, last line of main text: *for* $3, 4, 5, \ldots$ *read* $2, 3, 4, \ldots.$

p. 225, under $n = 45$, $r = 9$: *for* 5894·77474 8 *read* 5894·77484 8.

p. 227, reference to SCHLÄFLI: *insert* footnote symbol, †, after 'Kontinuen'.

THE DESIGN OF AN EXPERIMENT IN WHICH CERTAIN TREATMENT ARRANGEMENTS ARE INADMISSIBLE

BY D. R. COX
Statistical Laboratory, University of Cambridge

1. INTRODUCTION

Consider an experiment in which the experimental units are arranged in *sets* of k units, a set corresponding, for example, to a single production run of an industrial process. Suppose also that the k units in each set are arranged in order corresponding to the first, second, etc., *period* of the set. If there are both systematic differences between periods and an appreciable component of variance between sets, an experiment to compare a number of treatments, using these experimental units, would ordinarily be arranged in a Latin or Youden square, chosen to eliminate differences between periods and sets. Suppose, however, that for practical reasons there is a restriction on the order of treatments within each set, such as that the level of the treatment must not decrease from one period to the next in a set. Then Latin and Youden squares cannot be used. This paper is concerned with designs for such a situation; the method of construction is described and designs are listed for a few special cases. I have not hesitated to suggest designs that are nearly optimum whenever the determination of the strictly optimum design would be a very lengthy process.

The particular application for which this work was done concerned the properties of alloys cast from high-purity metals. Because of the high cost of the raw materials it was especially important to use an efficient experimental design. From a 20 lb. melt it was possible to cast four 5 lb. ingots, the smallest size convenient for further processing before testing. (In the language of the first paragraph the melt is a set, and the four ingots are the periods within a set, so that $k = 4$.) After an ingot had been cast, the composition of the remaining molten metal could be changed only by the *addition* of one of the alloying metals, no technique being available for selectively reducing the concentration of an alloying element. Therefore we have just the situation described above, in which the level of treatment cannot decrease from one period to the next in a set.

A further similar application is to certain large-scale experiments in steel-works, in which it is possible to tap several ladles from one cast of an open-hearth furnace. The effect of the addition of several elements could, with the present designs, be investigated by varying the chemical composition in successive ladles, rather than by the more costly and cumbersome procedure of varying the composition from cast to cast.

2. A SIMPLE CASE

Suppose that we wish to compare two treatments, A_{-1} and A_1, that $k = 4$ and that if A_1 occurs in one period of a set, all subsequent periods of that set must carry A_1. The only admissible sequences are:

Type	Period 1	Period 2	Period 3	Period 4
1	A_{-1}	A_{-1}	A_{-1}	A_{-1}
2	A_{-1}	A_{-1}	A_{-1}	A_1
3	A_{-1}	A_{-1}	A_1	A_1
4	A_{-1}	A_1	A_1	A_1
5	A_1	A_1	A_1	A_1

and the most general design is formed by taking n_i sets of type i, with $n_1 + \ldots + n_5 = N$ sets in all. Note that the treatment in period 1 is usually A_{-1} and in period 4 is usually A_1. Therefore some confounding of the treatment differences with periods is inevitable, unless only sequences 1 and 5 are used, in which case treatment differences are completely confounded with sets. Similarly some confounding with sets is inevitable, unless treatment differences are completely confounded with periods.

It will be assumed that the analysis of the results of such an experiment can be based on a linear set-up, in which any observation, y, is

(general mean) + (const. for set) + (const. for period)
+ (const. for treatment) + (random term),

where the random terms are uncorrelated and have zero mean and variance σ_r^2.

We shall first suppose that differences between sets and periods are to be eliminated completely. The best linear unbiased estimate, $\hat{d}^{(r)}$, of the treatment effect is obtained by the method of least squares and has variance $\lambda\sigma_r^2$, where λ depends only on the design, i.e. on $n_1, \ldots, n_5$. The best design is that which minimizes λ for given N.

To do the minimization, it is convenient to consider the problem as one of analysis of covariance. Introduce a concomitant variable, x, taking values -1 whenever A_{-1} is applied and 1 whenever A_1 is applied. The analysis of covariance of y and x, regarding the experiment as a two-way (set × period) arrangement, is

	s.s. of y	s.p. of x, y	s.s. of x
Between sets	$S(y,y)$	$S(x,y)$	$S(x,x)$
Between periods	$P(y,y)$	$P(x,y)$	$P(x,x)$
Residual	$R(y,y)$	$R(x,y)$	$R(x,x)$
Total	$T(y,y)$	$T(x,y)$	$T(x,x)$

By the theory of analysis of covariance, the residual regression coefficient $R(x,y)/R(x,x)$ is $\frac{1}{2}\hat{d}^{(r)}$, and by the formula for the variance of a regression coefficient

$$\text{var}\,(\tfrac{1}{2}\hat{d}^{(r)}) = \frac{\sigma_r^2}{R(x,x)}. \tag{1}$$

Thus we require to maximize $R(x,x)$ given N. First, it is clear by symmetry that for an optimum design $n_1 = n_5$, $n_2 = n_4$. Therefore

$$T(x,x) = 4N = 4(2n_1 + 2n_2 + n_3) \quad \text{and} \quad S(x,x) = \tfrac{1}{4}(2n_1 . 16 + 2n_2 . 4 + n_3 . 0),$$

since there are, for example, n_1 runs each with a total of -4 and n_1 each with a total of 4. We can find $P(x,x)$ similarly and thus get $R(x,x)$ by subtraction. The answer is

$$NR(x,x) = 12n_1n_2 + 8n_1n_3 + 6n_2n_3 + 4n_2^2. \tag{2}$$

Now maximize $NR(x,x)$ for fixed N introducing a Lagrange multiplier μ. The solution of the resulting linear equations is $n_1 = \frac{1}{5}\mu$, $n_2 = n_3 = \frac{3}{5}\mu$.

The simplest case in which these equations have a solution in integers is $n_1 = 1, n_2 = n_3 = 2$ and hence the simplest design minimizing the variance per observation is

$$\left.\begin{array}{cccc} A_{-1} & A_{-1} & A_{-1} & A_{-1} \\ A_{-1} & A_{-1} & A_{-1} & A_1 \\ A_{-1} & A_{-1} & A_{-1} & A_1 \\ A_{-1} & A_{-1} & A_1 & A_1 \\ A_{-1} & A_{-1} & A_1 & A_1 \\ A_{-1} & A_1 & A_1 & A_1 \\ A_{-1} & A_1 & A_1 & A_1 \\ A_1 & A_1 & A_1 & A_1 \end{array}\right\} \tag{3}$$

To use the design the order of the sets should be randomized.

Now $R(x,x) = 10$ so that $\quad \text{var}\,(\hat{d}^{(r)}) = \frac{2}{5}\sigma_r^2, \tag{4}$

and σ_r^2 may be estimated from the residual mean square of y adjusting for regression on x, i.e. by the formula

$$\hat{\sigma}_r^2 = \frac{1}{20}\left[R(y,y) - \frac{R^2(x,y)}{R(x,x)}\right]. \tag{5}$$

There are a number of designs with which (4) may be compared:

(*a*) if the treatments are randomized over eight sets subject only to the requirement that each treatment occurs twice in each set and four times in each period, the variance of the estimate of the treatment effect is $\frac{1}{8}\sigma_r^2$; this design does not satisfy our basic requirement but may be considered for comparison;

(*b*) if the component of variance between sets is σ_1^2, the variance of the mean observation on a set is $\sigma_1^2 + \frac{1}{4}\sigma_r^2$, so that if four sets are done with constant treatment A_{-1} and four with constant treatment A_1, the variance of the estimate is $\frac{1}{2}\sigma_1^2 + \frac{1}{8}\sigma_r^2$.

This shows that if the design (3) and the estimate $\hat{d}^{(r)}$ are used when $\sigma_1^2 \ll \sigma_r^2$, there is an appreciable loss of accuracy compared with the simple design in which the treatment is kept constant in each set. This loss arises from the non-orthogonality with periods and sets; the period differences are systematic, so that information confounded with them is lost, but set differences may, after randomization, be regarded as additional random variation. Hence a second estimate $\hat{d}^{(s)}$, uncorrelated with $\hat{d}^{(r)}$, may be derived from the regression coefficient between sets in the analysis of covariance. Thus

$$\tfrac{1}{2}\hat{d}^{(s)} = \frac{S(x,y)}{S(x,x)}, \tag{6}$$

$$\text{var}\,(\tfrac{1}{2}\hat{d}^{(s)}) = \frac{\sigma_s^2}{S(x,x)} = \frac{\sigma_s^2}{12}, \tag{7}$$

where $\sigma_s^2 = \sigma_r^2 + 4\sigma_1^2$ is the variance of the set totals on a unit basis and is estimated by

$$\hat{\sigma}_s^2 = \frac{1}{6}\left[S(y,y) - \frac{S^2(x,y)}{S(x,x)}\right]. \tag{8}$$

We combine $\hat{d}^{(r)}$ and $\hat{d}^{(s)}$ into a single estimate $\hat{d}$, where

$$\hat{d} = \left(\frac{R(x,x)}{\hat{\sigma}_r^2}\hat{d}^{(r)} + \frac{S(x,x)}{\hat{\sigma}_s^2}\hat{d}^{(s)}\right)\left(\frac{R(x,x)}{\hat{\sigma}_r^2} + \frac{S(x,x)}{\hat{\sigma}_s^2}\right)^{-1}, \tag{9}$$

$$\text{var}\,(\hat{d}) = 4\left(\frac{R(x,x)}{\sigma_r^2} + \frac{S(x,x)}{\sigma_s^2}\right)^{-1} = \frac{2\sigma_r^2(\sigma_r^2 + 4\sigma_1^2)}{11\sigma_r^2 + 20\sigma_1^2}, \tag{10}$$

neglecting errors in weighting.

In particular when $\sigma_1^2 = 0$, $\text{var}(\hat{d}) = \frac{3}{11}\sigma_r^2$, showing a substantial reduction from (4), and indicating that recovery of information is a worth-while process in this case. Also $\text{var}(\hat{d})$ is less than $\frac{1}{8}\sigma_r^2 + \frac{1}{8}\sigma_1^2$, the variance in design (*b*) above, when $\sigma_1^2/\sigma_r^2 > \frac{1}{4}$.

To sum up, the design (3) should be used only when an appreciable component of variance between sets is expected and information contained in the set totals should be recovered unless $\sigma_r^2 \ll \sigma_1^2$.

If practical conditions demand the use of a number of sets other than 8, or a multiple of 8, (2) may be used to determine the best values of n_1, n_2 and n_3.

Optimum designs similar to (3) may be found for other values of k. Two examples are

$k=2$		$k=3$		
A_{-1}	A_{-1}	A_{-1}	A_{-1}	A_{-1}
A_{-1}	A_1	A_{-1}	A_{-1}	A_{-1}
A_{-1}	A_1	A_{-1}	A_{-1}	A_1
A_1	A_1	A_{-1}	A_1	A_1
		A_1	A_1	A_1
		A_1	A_1	A_1

In the preceding analysis it has been assumed that two fixed treatments are to be compared. If, however, the linear regression on a factor that may be varied continuously is required, the concomitant variable, x, is no longer confined to two values. If, for practical reasons, the level of the factor must lie between an upper and a lower limit, these may be made to correspond to $x = \pm 1$; any admissible set of k treatments is then defined by a non-decreasing sequence of k terms, all between ± 1. Although I have not obtained a formal proof, it is probably true that so long as only the linear effect is required, the designs in which only the lower and upper levels appear are still optimum.

3. A 2^3 FACTORIAL DESIGN

As an example of the design of a more complicated experiment, consider a 2^3 factorial experiment in sets with four periods per set, assuming that the three-factor interaction is unimportant and that the level of each factor must not decrease from period to period within a set. Introduce six concomitant variables x_1, x_2, x_3, x_4 $(= x_{23})$, x_5 $(= x_{31})$ and x_6 $(= x_{12})$ to represent the main effects A, B, C and the two-factor interactions BC, CA, AB. Thus x_2 takes values ± 1 according as B is at its upper or its lower level and x_5 equals 1 for the combinations $A_{-1}C_{-1}$, A_1C_1 and -1 for the combinations $A_{-1}C_1$, A_1C_{-1}. Twice the residual regression coefficient of y on x_i is thus $\hat{a}_i^{(r)}$, the least-squares estimate of the corresponding main effect or interaction.

Let R_{ij} denote the residual sum of products of x_i and x_j in the analysis of covariance and $R_i(y)$ the residual sum of products of x_i and y. Then the column $\hat{\mathbf{a}}^{(r)}$ of estimates, and their variance-covariance matrix, are given by

$$\mathbf{R}\hat{\mathbf{a}}^{(r)} = 2\mathbf{R}(y), \tag{11}$$

$$\text{var}\,(\hat{\mathbf{a}}^{(r)}) = 4\mathbf{R}^{-1}\sigma_r^2. \tag{12}$$

We have now to choose the design to satisfy conditions on $\mathbf{R}^{-1}$; in particular we require the diagonal elements of $\mathbf{R}^{-1}$ to be small, especially those associated with the main effects. In a design in eight sets with one factor, (4) gives $\text{var}\,(\hat{a}^{(r)}) = \frac{2}{5}\sigma_r^2$. If several factors are investigated simultaneously the variance of a main effect cannot be less than the minimum

variance obtained when that factor is investigated by itself. Therefore if we can find a design in eight sets for which the variance of a main effect is inappreciably greater than $\frac{3}{8}\sigma_r^2$, this design is effectively optimum for the estimation of main effects.

A common-sense procedure for eight runs is to arrange that each factor separately has the design (3) and that the factor combinations are distributed as symmetrically as possible. This gives the design

$$\left.\begin{array}{cccc} A_{-1}B_1C_{-1} & A_{-1}B_1C_{-1} & A_{-1}B_1C_1 & A_{-1}B_1C_1 \\ A_{-1}B_{-1}C_{-1} & A_{-1}B_1C_1 & A_{-1}B_1C_1 & A_1B_1C_1 \\ A_{-1}B_{-1}C_{-1} & A_{-1}B_1C_{-1} & A_{-1}B_1C_{-1} & A_1B_1C_1 \\ A_{-1}B_{-1}C_{-1} & A_{-1}B_{-1}C_{-1} & A_1B_1C_{-1} & A_1B_1C_{-1} \end{array}\right\} \qquad (13)$$

$$\left.\begin{array}{cccc} A_{-1}B_{-1}C_1 & A_{-1}B_{-1}C_1 & A_1B_1C_1 & A_1B_1C_1 \\ A_{-1}B_{-1}C_{-1} & A_1B_{-1}C_{-1} & A_1B_{-1}C_{-1} & A_1B_1C_1 \\ A_{-1}B_{-1}C_{-1} & A_1B_{-1}C_1 & A_1B_{-1}C_1 & A_1B_1C_1 \\ A_1B_{-1}C_{-1} & A_1B_{-1}C_{-1} & A_1B_{-1}C_1 & A_1B_{-1}C_1 \end{array}\right\}$$

The matrix $\mathbf{R}$ of residual sums of products can be found easily. Main effects are orthogonal to interactions, i.e. the residual sum of products of any one of x_1, x_2, x_3 with any one of x_4, x_5, x_6 is zero. Thus $\mathbf{R}$ can be partitioned into two 3×3 matrices whose inverses may be found separately, i.e.

$$\mathbf{R} = \left(\begin{array}{c|c} \mathbf{R}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}_2 \end{array}\right), \qquad \mathbf{R}^{-1} = \left(\begin{array}{c|c} \mathbf{R}_1^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}_2^{-1} \end{array}\right),$$

where

$$\mathbf{R}_1 = \begin{pmatrix} 10 & 2 & -2 \\ 2 & 10 & -2 \\ -2 & -2 & 10 \end{pmatrix}, \quad \mathbf{R}_1^{-1} = \frac{1}{56}\begin{pmatrix} 6 & -1 & 1 \\ -1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix},$$

$$\mathbf{R}_2 = \begin{pmatrix} 22 & -2 & 4 \\ -2 & 22 & 4 \\ 4 & 4 & 8 \end{pmatrix}, \quad \mathbf{R}_2^{-1} = \frac{1}{96}\begin{pmatrix} 5 & 1 & -3 \\ 1 & 5 & -3 \\ -3 & -3 & 15 \end{pmatrix}. \qquad (14)$$

Thus the variance of the estimate of a main effect is $\frac{24}{56}\sigma_r^2 = \frac{3}{7}\sigma_r^2$, which is inappreciably greater than $\frac{3}{8}\sigma_r^2$, the variance in a single-factor design. Hence (13) is effectively optimum for estimating main effects.

The recovery of information between sets will not be considered in detail, because with one replicate of the design only a single degree of freedom is available for estimating σ_s^2. If several replicates are done, recovery of information should be considered. From the set totals in (13) estimates may be formed of $a_1 - a_2$, a_3 but not of a_1, a_2 separately. Hence if (13) is repeated the names of the treatments should be changed.

With eight sets an alternative arrangement to (13) is with the treatments held constant in each set, each of the eight combinations in the 2^3 experiment occurring once. The variance of the estimate of a main effect is then $\frac{1}{2}(\sigma_1^2 + \frac{1}{4}\sigma_r^2)$. Therefore (13) is the more accurate only if

$$\frac{3}{7}\sigma_r^2 < \tfrac{1}{2}\sigma_1^2 + \tfrac{1}{8}\sigma_r^2,$$

i.e. if

$$\sigma_1^2 > \tfrac{17}{28}\sigma_r^2. \qquad (15)$$

4. The estimation of curvatures

If it is required to estimate the curvature of the response to a factor in addition to the linear effect, we may introduce two concomitant variables for each main effect, the first, x', representing the linear component and the second, x'', the curvature. Thus with a factor at three equally spaced levels A_{-1}, A_0, A_1, x' takes values -1, 0, 1 and x'' values 1, -2, 1. Unless k and the number of factors are very small the number of possible designs is large and a formal mathematical analysis to determine the best design is rarely practicable.

However, when k is even we can write down a pair of sets from which the curvature may be estimated directly. Thus for $k = 4$

$$\left.\begin{array}{cccc} A_{-1} & A_{-1} & A_0 & A_0 \\ A_0 & A_0 & A_1 & A_1 \end{array}\right\} \tag{16}$$

gives an estimate of curvature from the difference of the observations on A_{-1}, A_1 and those on A_0. This estimate is clearly unaffected by period and set differences. Further, if a second factor B is superimposed on (16) according to the pattern

$$\left.\begin{array}{cccc} B_{-1} & B_1 & B_1 & B_1 \\ B_{-1} & B_{-1} & B_{-1} & B_1 \end{array}\right\} \tag{17}$$

the estimate of the curvature of A is unaffected by the main effect of B and by the linear-linear component of the interaction AB.

These results can be used in various ways by building up more complicated designs from the above pairs of sets. For example the following design in four sets is formed by interchanging A and B in the second pair:

$$\left.\begin{array}{cccc} A_{-1}B_{-1} & A_{-1}B_1 & A_0B_1 & A_0B_1 \\ A_0B_{-1} & A_0B_{-1} & A_1B_{-1} & A_1B_1 \\ A_{-1}B_{-1} & A_1B_{-1} & A_1B_0 & A_1B_0 \\ A_{-1}B_0 & A_{-1}B_0 & A_{-1}B_1 & A_1B_1 \end{array}\right\}. \tag{18}$$

If this design is analysed neglecting all interactions except the linear-linear one, the variance of a linear main effect is $\frac{1}{8}\sigma_r^2$, of the curvature of a main effect (defined as the regression coefficient of the response on x'') is $\frac{9}{160}\sigma_r^2$ and of the linear-linear interaction is $\frac{1}{4}\sigma_r^2$. Apart from the two curvatures, which are very slightly correlated, all estimates are independent. The most accurate estimate of the linear effect obtained in (3) with eight sets has a variance $\frac{1}{16}\sigma_r^2$ and hence the minimum variance in four sets is $\frac{1}{8}\sigma_r^2$. The increase in variance due to the inclusion of an estimate of curvature is not as large as might be expected. The accuracy of the linear effect relative to that of the curvature may be increased by the addition to (18) of sets derived from (3).

Another possibility is to introduce into the 2^3 experiment, (13), two new factors D and E whose curvature only is to be investigated. Hold D and E at their middle levels except in sets 2, 3, 6 and 7, and in these sets vary them in the following way:

$$\left.\begin{array}{lcccc} \text{Set } 2 & D_{-1}E_0 & D_{-1}E_0 & D_0E_0 & D_0E_0 \\ 3 & D_0E_0 & D_0E_0 & D_0E_1 & D_0E_1 \\ 6 & D_0E_0 & D_0E_0 & D_1E_0 & D_1E_0 \\ 7 & D_0E_{-1} & D_0E_{-1} & D_0E_0 & D_0E_0 \end{array}\right\}. \tag{19}$$

There is now a small amount of confounding with the interactions of A, B, C, but the loss of information is slight.

The number of possible designs is very large, and it is impossible to give a thorough account in a general paper such as this. The method is to decide how many factors are to be investigated, the number of sets available and the relative importance of linear effects and curvatures. Then build up from basic designs, such as (17), an arrangement with as much 'balance' as possible, and work out the matrix of the variances of the estimates. Next compare with simple designs such as (3) to see that too much accuracy has not been lost by the inclusion of extra factors. With practice this can be done fairly quickly, but clearly the work is only worth-while when economy in the design is of great importance.

5. Analysis of observations

The analysis of the observations from one of the above designs is essentially a straightforward process of multiple analysis of covariance, with the simplification that the matrix of sums of squares and products of the concomitant variables usually takes a simple form and can always be obtained and inverted beforehand. Also the concomitant variables are restricted to small integral values and sums of products with them can be formed rapidly.

Table 1 gives the analysis for the 2^3 design (13). Fictitious observations have been constructed by taking the main effects of A, B, C to be $2, -2, 1$ and the interactions to be zero. The uncontrolled variation has been represented by period differences, by a random variation between sets of standard deviation 2, and by a random variation from unit to unit of standard deviation 1. A constant has been added to make all the observations positive.

Figures in bold type are constants for the design and may be entered before the observations are obtained. The procedure should be clear from the following notes:

(1) First form the analysis of variance of the observations, y. into correction, between sets (rows), between periods (columns) and residual.

(2) Then find the total sums of products of y with the six concomitant variables. For example x_{23} takes values 1 for B_1C_1 and $B_{-1}C_{-1}$ and -1 otherwise. It is desirable to form the total sum of products separately by rows and by columns to obtain a check. Thus, by rows,

$$\Sigma y x_{23} = -3{\cdot}92 - 6{\cdot}44 + 7{\cdot}87 + \ldots - 6{\cdot}92 - 7{\cdot}35 = 48{\cdot}66.$$

(3) The sum of products for sets is obtained from the sum of products of the set totals of y and the concomitant variable. Thus for x_{23} it is

$$\tfrac{1}{4}(20{\cdot}36 \times 4 + 27{\cdot}50 \times 4) = 47{\cdot}86.$$

Similarly for periods. In this way the full analysis of covariance table is obtained, calculating the residual term as total + correction − rows − columns, since the individual terms have not be corrected.

(4) The estimates of the regression coefficients are formed by multiplying the residual sums of products by the appropriate coefficients, obtained from (14), given in bold type. Thus the regression coefficient on x_{31} is

$$\tfrac{1}{96}[1 \times (-1{\cdot}035) + 5 \times (-8{\cdot}605) + (-3) \times (-2{\cdot}06)] = -0{\cdot}3946.$$

The sum of squares for regression is obtained as the sum of products of the regression coefficients with the corresponding residual sum of products, and may, in this instance, be collected separately for main effects and for interactions. Thus, for main effects, the sum of squares is

$$1{\cdot}2895 \times 9{\cdot}705 + (-1{\cdot}4718) \times (-12{\cdot}385) + 0{\cdot}1230 \times 1{\cdot}595 = 30{\cdot}9390.$$

This leads to an analysis of variance from which the error variance is estimated by 1·3845 with 15 degrees of freedom (the population value is 1).

Table 1. *Analysis of 2^3 experiment*

$A_{-1}B_1C_{-1}$	3·92	$A_{-1}B_1C_{-1}$	6·44	$A_{-1}B_1C_1$	7·87	$A_{-1}B_1C_1$	9·81
$A_{-1}B_{-1}C_{-1}$	5·13	$A_{-1}B_1C_1$	2·99	$A_{-1}B_1C_1$	3·87	$A_1B_1C_1$	8·37
$A_{-1}B_{-1}C_{-1}$	4·10	$A_{-1}B_1C_{-1}$	1·89	$A_{-1}B_1C_{-1}$	5·27	$A_1B_1C_1$	8·27
$A_{-1}B_{-1}C_{-1}$	7·07	$A_{-1}B_{-1}C_{-1}$	5·81	$A_1B_1C_{-1}$	9·22	$A_1B_1C_{-1}$	8·80
$A_{-1}B_{-1}C_1$	10·78	$A_{-1}B_{-1}C_1$	9·01	$A_1B_1C_1$	9·01	$A_1B_1C_1$	12·13
$A_{-1}B_{-1}C_{-1}$	5·66	$A_1B_{-1}C_{-1}$	7·86	$A_1B_{-1}C_{-1}$	8·35	$A_1B_1C_1$	5·63
$A_{-1}B_{-1}C_{-1}$	2·97	$A_1B_{-1}C_1$	7·29	$A_1B_{-1}C_1$	8·68	$A_1B_1C_1$	7·68
$A_1B_{-1}C_{-1}$	5·90	$A_1B_{-1}C_{-1}$	5·75	$A_1B_{-1}C_1$	6·92	$A_1B_{-1}C_1$	7·35

Set (row) totals

Set	y	x_1	x_2	x_3	x_{23}	x_{31}	x_{12}
1	28·04	−4	4	0	0	0	−4
2	20·36	−2	2	2	4	0	0
3	19·53	−2	2	−2	0	4	0
4	30·90	0	0	−4	0	0	4
5	40·93	0	0	4	0	0	4
6	27·50	2	−2	−2	4	0	0
7	26·62	2	−2	2	0	4	0
8	25·92	4	−4	0	0	0	−4
	219·80	0	0	0	8	8	0

Period (column) totals

Period	y	x_1	x_2	x_3	x_{23}	x_{31}	x_{12}
1	45·53	−6	−6	−6	4	4	4
2	47·04	−2	−2	−2	0	0	−4
3	59·19	2	2	2	0	0	−4
4	68·04	6	6	6	4	4	4
	219·80	0	0	0	8	8	0

Analysis of covariance

	y^2	yx_1	yx_2	yx_3	yx_{23}	yx_{31}	yx_{12}	D.F.
Correction	1509·7512	0	0	0	54·95	54·95	0	1
Sets	1587·2470	4·995	−4·995	10·005	47·86	46·15	17·87	8
Periods	1552·3300	19·92	19·92	19·92	56·785	56·785	3·67	4
Residual	55·2266	9·705	−12·385	1·595	−1·035	−8·605	−2·06	21
Total*	1685·0524	34·62	2·54	31·52	48·66	39·38	19·48	32

* Checked by columns and by rows. Note that terms for sets and periods are uncorrected.

$$\frac{1}{56}\begin{pmatrix} 6 & -1 & 1 \\ -1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix} \qquad \frac{1}{96}\begin{pmatrix} 5 & 1 & -3 \\ 1 & 5 & -3 \\ -3 & -3 & 15 \end{pmatrix}$$

Regr. coeff.	1·2895	−1·4718	0·1230	−0·0792	−0·3946	−0·0206
Contr. to s.s.	12·5146	18·2282	0·1962	0·0820	3·3955	0·0424
Variance	0·1483	0·1483	0·1483	0·0721	0·0721	0·2163
St. error	0·3851	0·3851	0·3851	0·2685	0·2685	0·4651

	D.F.	y^2	Mean square
Regr. on main effects (x_1, x_2, x_3)	3	30·9390	10·3130
interactions (x_{23}, x_{31}, x_{12})	3	3·5199	1·1733
Error	15	20·7677	1·3845
Residual (uncorrected for regr.)	21	55·2266	—

	Main effects			Interactions		
	A	*B*	*C*	*BC*	*CA*	*AB*
Estimate	2·5790	−2·9436	0·2460	−0·1584	−0·7892	−0·0412
St. error	0·7702	0·7702	0·7702	0·5370	0·5370	0·9302
True value	2	−2	1	0	0	0

(5) The variances of the regression coefficients are obtained by multiplying the error variance by the appropriate diagonal element of the matrices of coefficients. Confidence limits (or significance tests) for an individual regression coefficient follow from the t distribution in the usual way.

(6) Main effects and interactions are twice the corresponding regression coefficients.

6. Discussion

We have considered experimental design in a special situation where practical conditions severely limit the treatment arrangements that may be used. The limitation makes a substantial amount of non-orthogonality inevitable if certain portions of the uncontrolled variation (set and period differences) are to be eliminated from the error.

The following remarks are relevant to the application of the designs developed above and of other designs that may be worked out for similar situations:

(i) The design of even a simple experiment to compare two treatments requires care.

(ii) The number of factors that may be examined simultaneously is restricted.

(iii) It is better not to attempt to eliminate set differences unless there is an appreciable component of variance between sets (§§ 2, 3).

(iv) The correct method of analysis must be used, but this is usually straightforward once the sequence of calculations has been grasped.

(v) Finally, it must be repeated that the situation is a complicated one and that the use of special designs, while sometimes very well worth while, is usually only justified when economy in the experimental material is of great importance.

I am very grateful to Dr C. J. Anson, G. K. N. Group Research Laboratory, for suggesting the problem and for providing the description of the application, and to Dr J. Wishart for helpful comments on the draft of the paper.

THE ESTIMATION OF LOCATION AND SCALE PARAMETERS FROM GROUPED DATA

By J. M. HAMMERSLEY† and K. W. MORTON‡

We shall describe a method of estimating the location and scale parameters of an arbitrary continuous distribution, when the data consist of the numbers of observations (in a sample) falling within arbitrary specified intervals of the variate. The method is closely allied to that given by Lloyd (1952) for estimating these parameters, when the data comprise order statistics; and such novelties as our method possesses consist only in (i) adopting a slightly different viewpoint of the data so that they may be looked upon as grouped rather than ordered, and (ii) introducing approximations which considerably simplify the numerical analysis. Mosteller (1946) discusses some related topics.

We found the method particularly suitable in the following circumstances, which we describe in general terms partly to help to fix the ideas and partly because similar demanding circumstances may arise in other contexts. In an experiment on the ionization loss of relativistic μ-mesons in neon (Eyeions *et al.* 1954) a sample of observations was taken under each of several experimental conditions. Theory suggested that, although the parent distributions of the different samples were all different and all of unknown functional form, they would differ only in being linear transforms of each other; and the object of the experiment was to discover how their location and scale parameters depended upon the experimental conditions. By taking a very large sample under fixed experimental conditions it was possible to get an empirical picture of the shape of the cumulative distribution function; but it was not practicable to take large samples under the remaining experimental conditions. Indeed, the data from the several small samples could only be gathered in irregular and relatively coarse groups. The shape of the distribution (found from the large sample) appeared to be very skew, observations in the tails of the distribution seemed so erratic that arbitrary truncation seemed advisable, and one of the tails was so long as to render estimating location by an arithmetic mean quite unreliable. Further, although we had a fair picture of the empirical cumulative distribution function from this large sample, it would barely stand one numerical differentiation to give a representation of the shape of the probability density function, and certainly not two differentiations. Hence any form of maximum-likelihood estimation was out of the question. Minimum χ^2 estimation was a possibility; but, particularly here where two parameters had to be estimated simultaneously for each of a number of small samples, the numerical labour of that method was discouragingly heavy; and, moreover, it was not obvious how to derive standard errors of the resulting estimates.

Let $F(x)$ denote the (true unknown) cumulative distribution function of the large sample; and let $F(\alpha+\beta x)$ be the corresponding cumulative distribution function of a given small sample. Here α and β are unknown parameters to be estimated. Suppose that the intervals, characterizing the grouping of the small sample, are $x_{i-1} < z \leqslant x_i$ $(i = 2, 3, \ldots, N)$. These

† Readership in the Design and Analysis of Scientific Experiment, University of Oxford; and Consultant, Atomic Energy Research Establishment, Harwell.

‡ Atomic Energy Research Establishment, Harwell.

x_i are known; and we shall suppose that (before truncating the sample, if truncation occurs) each group contains at least one observation, and in particular that there is at least one observation less than or equal to x_1 and at least one observation greater than x_N. This requirement can always be met by amalgamating groups if necessary. Suppose that the small sample contains m observations *before* truncation (if that is necessary); and that, of these m observations, m_i do not exceed x_i. Define $p_i = m_i/m$; and let y_i be the solution of $F^*(y_i) = p_i$, where $F^*(x)$ is the observed empirical cumulative distribution function of the large sample. (If the functional form of F happens to be known, we may of course write F for F^* in all that follows.) In virtue of the foregoing suppositions we have

$$0 < p_1 < p_2 < \ldots < p_N < 1.$$

We now make two approximations: (i) we suppose that the large sample gives a sufficiently good representation of its cumulative distribution function (though the corresponding probability density function may not be known nearly so well) to allow us to write $F^*(y_i) = F(y_i)$, and (ii) we suppose that in the immediate neighbourhood, over which y_i may vary due to sampling effects, F is approximately linear, so that we may write

$$\mathscr{E}[F(y_i)] = F[\mathscr{E}(y_i)].$$

Then we have

$$F(\alpha+\beta x_i) = \mathscr{E}(p_i) = \mathscr{E}[F^*(y_i)] = \mathscr{E}[F(y_i)] = F[\mathscr{E}(y_i)],$$

and hence

$$\mathscr{E}(y_i) = \alpha + \beta x_i. \tag{1}$$

The method proposed now is to estimate α and β by finding the linear regression of y_i on x_i.

In carrying through this linear regression we must remember that the y_i are correlated (say with variance-covariance matrix $\mathbf{V}$), and that they will have to be weighted accordingly. The weights we shall obtain depend upon the first derivative of $F(x)$; and hence will be somewhat inaccurate; but here, and also in finding the standard errors of α and β, we can accept a lower standard of accuracy than was needed in deriving (1).

It is a standard result (Frisch, 1928; Aitken, 1935) that the unbiased minimum variance linear estimators a and b of α and β are given by

$$\begin{pmatrix} a \\ b \end{pmatrix} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}, \tag{2}$$

where

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \end{pmatrix}, \quad \mathbf{y}' = (y_1, y_2, \ldots, y_N),$$

and that the sampling variance-covariance matrix of these estimates is $(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$. We proceed to calculate $\mathbf{V}$ and $\mathbf{V}^{-1}$. Since we are assuming that F is approximately linear in the neighbourhood of y_i, the elements of $\mathbf{V}$ are $v_{ij} = \operatorname{cov}(y_i, y_j) = \operatorname{cov}(p_i, p_j)/f_i f_j$, where $f_i = [\partial F^*(x)/\partial x]_{x=y_i}$. Now m_j is the number of 'successes' in a sample of m where the probability of success is $\mathscr{E}(p_j)$; and hence approximately $\operatorname{var} m_j = mp_jq_j$, where $q_j = 1-p_j$. If $i<j$ and m_j is given, then m_i is the number of successes in a sample of m_j where the probability of success is $\mathscr{E}(p_i)/\mathscr{E}(p_j)$; and hence approximately

$$\mathscr{E}(m_i \mid m_j) = \frac{\operatorname{cov}(m_i, m_j)}{\operatorname{var} m_j} m_j = \frac{p_i}{p_j} m_j.$$

Thus $\operatorname{cov}(m_i, m_j) = mp_iq_j$ for $i<j$; and therefore for $i \leqslant j$ by comparison with the result for $\operatorname{var} m_j$. It follows that $\operatorname{cov}(p_i, p_j) = p_iq_j/m$ for $i \leqslant j$; and so

$$v_{ji} = v_{ij} = p_iq_j/mf_if_j \quad (i \leqslant j). \tag{3}$$

By constructing the product $\mathbf{V}\mathbf{V}^{-1}$, it can be verified that

$$\mathbf{V}^{-1} = m \begin{pmatrix} c_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \gamma_1 & c_2 & \gamma_2 & \dots & 0 & 0 \\ 0 & \gamma_2 & c_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_{N-1} & \gamma_{N-1} \\ 0 & 0 & 0 & \dots & \gamma_{N-1} & c_N \end{pmatrix},$$

where (with the conventions $p_0 = 0, p_{N+1} = 1$)

$$c_i = f_i^2(p_{i+1} - p_{i-1})/(p_{i+1} - p_i)(p_i - p_{i-1}), \quad \gamma_i = -f_i f_{i+1}/(p_{i+1} - p_i). \tag{4}$$

The approximations which have led up to (3) and hence allowed the explicit inversion of V provide the main simplification in the numerical analysis, mentioned at the outset. It is now easy to complete the estimation by substitution into (2). We collect the results in the form of a computing recipe:

I. Compute $p_i = m_i/m$ for $i = 1, 2, \dots, N$; hence determine y_i such that $F^*(y_i) = p_i$, and thence $f_i = [\partial F^*(x)/\partial x]_{x=y_i}$.

II. Compute c_i $(i = 1, 2, \dots, N)$ and γ_i $(i = 1, 2, \dots, N-1)$ from (4), where $p_0 = 0$, $p_{N+1} = 1$.

III. Compute

$$S_{11} = \sum_{i=1}^{N} c_i + 2\sum_{i=1}^{N-1} \gamma_i, \quad S_{1x} = \sum_{i=1}^{N} c_i x_i + \sum_{i=1}^{N-1} \gamma_i (x_i + x_{i+1}),$$

$$S_{xx} = \sum_{i=1}^{N} c_i x_i^2 + 2\sum_{i=1}^{N-1} \gamma_i x_i x_{i+1}, \quad S_{1y} = \sum_{i=1}^{N} c_i y_i + \sum_{i=1}^{N-1} \gamma_i (y_i + y_{i+1}),$$

$$S_{xy} = \sum_{i=1}^{N} c_i x_i y_i + \sum_{i=1}^{N-1} \gamma_i (x_i y_{i+1} + x_{i+1} y_i), \quad D = S_{11} S_{xx} - S_{1x}^2.$$

IV. Then the estimates of α and β are

$$a = (S_{xx} S_{1y} - S_{1x} S_{xy})/D, \quad b = (S_{11} S_{xy} - S_{1x} S_{1y})/D,$$

with sampling variances and covariance

$$\operatorname{var} a = S_{xx}/mD, \quad \operatorname{var} b = S_{11}/mD, \quad \operatorname{cov}(a, b) = -S_{1x}/mD.$$

To exhibit the analogy between the foregoing analysis and Lloyd's work, we conclude by generalizing Lloyd's argument and simplifying the appropriate numerical work. In contrast with our previous suppositions we now suppose that, when m is the size of the small sample, m_i $(i = 1, 2, \dots, N)$ is a *prescribed* set of integers satisfying $0 < m_1 < m_2 < \dots < m_N \leqslant m$; and we write ξ_i for the m_ith observation when the observations are arranged in ascending order of magnitude. Suppose that the available data from the small sample consists of the values ξ_i $(i = 1, 2, \dots, N)$ and *only* these values. (Lloyd considered the natural but nevertheless special case $N = m$.) Instead of p_i we now write $\pi_i = m_i/(m+1)$; and we define η_i as the solution of $F_i^*(\eta_i) = \pi_i$. Since $F(\alpha + \beta\xi_i)$ is the m_ith observation in an ordered sample of m from a rectangular distribution we have

$$\mathscr{E}[F(\alpha + \beta\xi_i)] = \pi_i, \quad \operatorname{cov}[F(\alpha + \beta\xi_i), F(\alpha + \beta\xi_j)] = \pi_i(1 - \pi_j)/(m+2) \quad (i \leqslant j). \tag{5}$$

Hence, with the same approximations as before,

$$\mathscr{E}(\xi_i) = (\eta_i - \alpha)/\beta, \quad \operatorname{cov}(\xi_i, \xi_j) = \pi_i(1-\pi_j)/(m+2)\beta^2\phi_i\phi_j \quad (i \leqslant j), \tag{6}$$

where $\phi_i = [\partial F^*(x)/\partial x]_{x=\eta_i}$. We now consider the regression of ξ_i upon η_i (instead of y_i upon x_i); and the analysis proceeds as before.

The resulting computing recipe becomes:

Ia. Compute $\pi_i = m_i/(m+1)$ for $i = 1, 2, \ldots, N$; and hence determine η_i such that $F^*(\eta_i) = \pi_i$, and thence $\phi_i = [\partial F^*(x)/\partial x]_{x=\eta_i}$.

IIa. Compute c_i $(i = 1, 2, \ldots, N)$ and γ_i $(i = 1, 2, \ldots, N-1)$ from (4) with π_i for p_i and ϕ_i for f_i, where $\pi_0 = 0$ and $\pi_{N+1} = 1$.

IIIa. Compute the quantities in III, writing η in place of x and ξ in place of y throughout.

IVa. Then the estimates of $-\alpha\beta^{-1}$ and β^{-1} are

$$-ab^{-1} = (S_{\eta\eta}S_{1\xi} - S_{1\eta}S_{\eta\xi})/D, \quad b^{-1} = (S_{11}S_{\eta\xi} - S_{1\eta}S_{1\xi})/D$$

with estimated sampling variances and covariances

$$\operatorname{var}(-ab^{-1}) = (b^{-1})^2 S_{\eta\eta}/(m+2)D, \quad \operatorname{var}(b^{-1}) = (b^{-1})^2 S_{11}/(m+2)D,$$
$$\operatorname{cov}(ab^{-1}, b^{-1}) = -(b^{-1})^2 S_{1\eta}/(m+2)D.$$

NUMERICAL EXAMPLE

We illustrate the computing recipe I, ..., IV by carrying through the numerical work for one of the small samples in the ionization loss problem.

Before considering the small samples, it is necessary to do a certain amount of computing on the large sample. This was presented as an empirical frequency function $f^*(y)$, tabulated at equal intervals of y. We are more concerned with comparing the small samples with each other than with setting an absolute standard. Hence it is advisable to smooth $f^*(y)$, which we do by means of its differences.

This is then integrated to give a table of $F^*(y)$, the proportion of observations not exceeding y. From this, a table of y against $F^*(y)$ at equal intervals of $F^*(y)$ is prepared, doing the inverse interpolation by iterating the formula

$$\theta = [F^*_\theta - F^*_0 - B^{\mathrm{II}}(\theta)(\delta^2_m F^*_0 + \delta^2_m F^*_1) - B^{\mathrm{III}}(\theta)\delta^3 F^*_{\frac{1}{2}}]/\delta F^*_{\frac{1}{2}},$$

as described by Hartree (1952, p. 91).

Numerically the large sample gives the results shown in Tables 1 and 2 in which $f(y)$ is a smoothed version of $f^*(y)$.

This now enables us to deal easily with each of the small samples. We have x_i and p_i from the data, and by interpolation in Table 2 we obtain y_i. From Table 1 we then get $f_i = f(y_i)$ and the computing recipe then gives us Table 3, where $m_0 = 0$ and $m_{N+1} = m$, by convention. Each m_i is derived from averaging over two counters, counting common events in parallel; this explains the occurrence of half units in the relevant column. Δp_i is inserted for convenience in calculating c_i and γ_i. Similarly, $c_i x_i$ and $\gamma_i x_i$ are tabulated for convenience in accumulating the sums

$$S_{11} = 0{\cdot}1945, \quad S_{1x} = -0{\cdot}1064, \quad S_{xx} = 1{\cdot}0410, \quad S_{1y} = 1{\cdot}2255, \quad S_{xy} = 0{\cdot}2326, \quad D = 0{\cdot}1912.$$

Thence

$$a = 6{\cdot}802, \quad b = 0{\cdot}919,$$
$$\operatorname{var} a = 0{\cdot}0058, \quad \operatorname{cov}(a, b) = 0{\cdot}0006, \quad \operatorname{var} b = 0{\cdot}0011.$$

Table 1

y	$f^*(y)$	$f(y)$	$F^*(y)$
0	0·0000	0·0000	0·0000
1	·0002	·0002	·0000
2	·0013	·0015	·0007
3	·0064	·0068	·0043
4	·0230	·0219	·0174
5	·0682	·0715	·0593
6	0·1484	0·1444	0·1699
7	·1756	·1745	·3338
8	·1461	·1510	·4992
9	·1127	·1116	·6310
10	·0732	·0728	·7223
11	·0506	·0503	·7828
12	0·0334	0·0349	0·8250
13	·0257	·0246	·8543
14	·0193	·0189	·8758
15	·0147	·0150	·8927
16	·0106	·0116	·9060
17	·0085	·0082	·9160

Table 2

$F^*(y)$	y	$F^*(y)$	y	$F^*(y)$	y
0	0	0·10	5·452	0·86	13·239
0·01	3·576	·15	5·859	·87	13·702
·02	4·113	·20	6·201	·88	14·226
·03	4·442	·25	6·512	·89	14·822
·04	4·678	·30	6·806	·90	15·515
·05	4·860	·35	7·093	·91	16·363
		0·40	7·384		
		·45	7·685		
		·50	8·005		
		·55	8·352		
		·60	8·735		
		·65	9·176		
		0·70	9·713		
		·75	10·412		
		·80	11·365		
		·85	12·829		

Table 3

i	x_i	m_i	p_i	$\Delta p_i = p_{i+1} - p_i$	y_i	f_i	c_i	$c_i x_i$	γ_i	$\gamma_i x_i$
0	−6	0	0	0·011	—	—	—	—	—	—
1	−3	10	0·011	0·044	3·643	0·0147	0·02448	−0·0734	−0·02209	0·0663
2	−2	52	·055	·103	4·930	·0662	·14219	− ·2844	− ·08986	·1797
3	−1	148	·158	·148	5·917	·1398	·32176	− ·3218	− ·16380	·1638
4	0	287	·306	·154	6·840	·1734	·39849	0	− ·17969	0
5	1	431·5	·460	·119	7·747	·1596	·37931	·3793	− ·17344	− ·1734
6	2	542·5	0·579	0·104	8·568	0·1293	0·30143	0·6029	−0·11212	−0·2242
7	3	640·5	·683	·068	9·516	·0901	·19766	·5930	− ·08172	− ·2452
8	4	704	·751	·050	10·427	·0616	·13187	·5274	− ·05398	− ·2159
9	5	750·5	·801	·032	11·386	·0438	·09828	·4914	− ·04380	− ·2190
10	6	780·5	·833	·038	12·234	·0320	·05898	·3539	− ·01686	− ·1012
11	7	816·5	0·871	0·021	13·751	0·0200	0·02960	0·2072	−0·01446	−0·1012
12	8	836	·892	·017	14·951	·0152	·02453	·1962	− ·00958	− ·0766
13	9	852	·909	·011	16·268	·0107	·01724	·1552	− ·00658	− ·0592
14	10	862·5	·920	·080	17·500	·0068	·00471	·0471	—	—
15 ($=N+1$)	∞	937·5 ($=m$)	1·000	—	—	—	—	—	—	—

To check the validity of this approach we carried out a χ^2 goodness-of-fit test as follows:

$$\chi^2 = m \sum_{i=0}^{N} (\Delta p_i - \Delta F_i^*)^2 / \Delta F_i^*,$$

where $\Delta p_i = p_{i+1} - p_i$ and $\Delta F_i^* = F^*(a + bx_{i+1}) - F^*(a + bx_i)$.

Numerically $\chi^2 = 12{\cdot}8$, and with 12 degrees of freedom this occurs near the 40 % point.

We are indebted to Dr J. Howlett, Atomic Energy Research Establishment, Harwell, for calling our attention to this problem, and to Mrs B. M. Sykes and Miss A. Beard of the Harwell computing group, for carrying out the computation in the numerical example.

REFERENCES

AITKEN, A. C. (1935). On least squares and linear combination of observations. *Proc. Roy. Soc. Edinb.* **55**, 42–48.

EYEIONS, D. A., OWEN, B. G., PRICE, B. T. & WILSON, J. G. (1954). The ionization loss of relativistic μ-mesons in neon. *Proc. Phys. Soc.* (to be published).

FRISCH, R. (1928). Correlation and scatter in statistical variables. *Nordisk Statist. Tidskr.* **8**, 36–102.

HARTREE, D. R. (1952). *Numerical Analysis*, 1st ed. Oxford University Press.

LLOYD, E. H. (1952). Least-squares estimation of location and scale parameters. *Biometrika*, **39**, 88–95.

MOSTELLER, F. (1946). On some useful 'inefficient' statistics. *Ann. Math. Statist.* **17**, 377–408.

TRANSFORMATIONS OF THE BINOMIAL, NEGATIVE BINOMIAL, POISSON AND χ^2 DISTRIBUTIONS

By GUNNAR BLOM

Department of Mathematical Statistics, University of Stockholm

1. Summary and introduction

Transformations of statistical variables are used in applied statistics mainly for two purposes.

(i) *Variance stabilization*

A transformation is applied in order to make it possible to use, at any rate approximately, the standard techniques associated with continuous normal variation, e.g. the methods of analysis of variance.

In particular, transformations are required which stabilize the variance, that is, which make the variance of the transformed variable approximately independent of, for example, the binomial probability or the mean value of the Poisson distribution. It is, of course, desirable that the transformed variable, satisfying the constant-variance condition, is also normally distributed. As all transformations, subject to mild restrictions, are asymptotically normal, this part of the transformation problem has in general been treated only superficially. The constant-variance condition has led to the introduction of the inverse sine, inverse sinh and square-root transformations which are used nowadays in many fields of applications (see the references given by Eisenhart (1947), which, together with those given at the end of this paper, constitute a fairly complete bibliography of this part of the subject.)

(ii) *Normalization*

A transformation is used in order to facilitate the computation of tail sums of the distribution by the aid of the normal probability integral. There are two problems in this connexion, which are here formulated in terms of the binomial distribution but apply *mutatis mutandis* to the other distributions mentioned in the title as well (excluding the confidence problem which does not apply to the χ^2-distribution).

The tail problem. Determination of the area of a tail of the binomial distribution; that is, calculation of the probability that the variable is greater than or equal to a given value a or less than or equal to a.

The confidence problem. Numerical determination of confidence limits for an unknown binomial probability p in terms of an observed value a.

The transformations introduced for variance stabilization have been applied by several authors to the last-mentioned problems, the reason for this being (i) the asymptotic normality, (ii) the satisfactory results of empirical trials (see, for example, Paulson, 1947; Hald, 1952). A modification of the inverse sine transformation, known as the chordal transformation, has been introduced by Mosteller & Tukey (1949). Freeman & Tukey (1941, 1950) have also made an empirical study of both the 'stabilization' and 'normalization' effects of various formulae founded upon the inverse sine and chordal transformations.

A review of the literature shows that a considerable number of transformations of binomial, negative binomial, Poisson and χ^2 variables have been proposed. In addition to those cited above, we may mention the logarithmic transformation which has been applied to the binomial and the χ^2 distributions, and the cube-root transformation, which has been used in connexion with the Poisson and χ^2 distributions.

In this paper, whose contents can be summarized as follows, we shall be concerned only with the tail problem and the confidence problem.

In § 5 we introduce a general transformation of the binomial distribution and expand the transformed variable in a Cornish-Fisher (C.F.) series (Cornish & Fisher, 1937). In this way general solutions of the two problems mentioned above in terms of the normal probability integral are obtained.

In § 6 the central idea of the paper is presented, namely, to derive a transformation of the binomial distribution that gives the C.F. expansion the property that the term of order $n^{-\frac{1}{2}}$ ('the skewness correction') is as small as possible. The application of this principle leads to a differential equation which contains the desired transformation as a solution. In the following sections the same idea is applied to the Poisson, χ^2 and negative binomial distributions. It will be shown that the differential equations obtained in this way are satisfied *inter alia* by the following transformations: inverse sine, square root, cube root and logarithmic. Thus, we shall find that nearly all the transformations used hitherto in the literature for normalization of binomial and related variables *can be developed from a common starting point.*

Some new transformations are also derived, which seem to be of a certain interest, at any rate from a theoretical point of view. For some transformations which are of more general interest, the C.F. series, furnishing solutions to the tail problem and the confidence problem, are presented. A numerical study of some of the formulae is made, and a comparison with standard formulae is included. As the main purpose of the paper is, however, not to derive very accurate approximation procedures, but to seek for general principles, no systematic comparison between the host of formulae which can be applied to the tail problem and the confidence problem has been made.

One of the practical conclusions of the investigations reported here is that the inverse sine transformation and its limiting case, the square-root transformation, may be used not only when stabilization of the variance is essential but also when normalization is the main purpose of the transformation procedure. Therefore the formulae founded upon these transformations may with advantage replace the classical large-sample methods for handling binomial and Poisson variables.

The derivation of the C.F. expansions used in the paper can be performed in different ways. An interesting line of approach is to utilize the relationship between the binomial, negative binomial and Poisson distributions on the one hand and Fisher's z-distribution on the other. This method of substituting a continuous distribution for a discontinuous one may seem artificial at first sight but is, as we shall see, valuable for two reasons: first, because no question of continuity corrections arises, and secondly, because the tail problem and the confidence problem can then be treated uniformly. The formulae describing this relationship are given in §§ 3 and 4.

Readers who are interested only in the results can begin with § 7 and consult the preceding sections when necessary.

2. General formulation of the problems

The two problems mentioned in the introduction can be formulated in compact form as follows.

Let x be a random variable. By $P(x \geqq a; \theta)$ we denote the probability that the variable is greater than or equal to a. θ is a parameter (known or unknown, as the case may be). $P(x \geqq a; \theta)$ is supposed to be a monotone function of θ for any fixed a. Put

$$P(x \geqq a; \theta) = \epsilon. \tag{2·1}$$

The tail problem arises, when ϵ is to be calculated, and the confidence problem, if θ is unknown. In the latter case $1-\epsilon$ is the confidence level (one-sided interval).

3. The binomial case

In particular, let x follow the binomial distribution with the parameters n and p. Further, let the variable y be beta-distributed with f_1 and f_2 degrees of freedom so that y has the frequency function

$$\text{const.}\, y^{\frac{1}{2}f_1-1}(1-y)^{\frac{1}{2}f_2-1}.$$

The two distributions are connected by means of the well-known formula

$$p^n + np^{n-1}q + \ldots + \binom{n}{a} p^a q^{n-a} = \frac{\Gamma(n+1)}{\Gamma(a)\,\Gamma(n-a+1)} \int_0^p y^{a-1}(1-y)^{n-a}\,dy, \tag{3·1}$$

or

$$P(x \geqq a; p) = P(y \leqq p; f_1, f_2), \tag{3·2}$$

where

$$f_1 = 2a, \quad f_2 = 2(n-a+1). \tag{3·3}$$

Using the fact that the variable $\dfrac{f_1 F}{f_1 F + f_2}$, where F has the F-distribution with f_1 and f_2 degrees of freedom, is beta-distributed with the same degrees of freedom, we can write

$$P(x \geqq a; p) = P\left(\frac{f_1 F}{f_1 F + f_2} \leqq p; f_1, f_2\right),$$

where f_1 and f_2 are given by (3·3). With

$$p^* = \frac{a}{n+1}, \quad q^* = 1-p^* = \frac{n-a+1}{n+1}, \tag{3·4}$$

we find

$$P(x \geqq a; p) = P\left(\frac{p^*}{q^*} F \leqq \frac{p}{q}\right) = P(z \leqq Z), \tag{3·5}$$

where z is z-distributed with the degrees of freedom given by (3·3) and

$$2Z = \log p/q - \log p^*/q^*. \tag{3·6}$$

It follows from these relations that the confidence problem for a binomial variable is identical with a percentage point problem for a z-variable. In addition, we have replaced a discontinuous variable by a continuous one, which has certain advantages, as previously mentioned.

4. The Cornish-Fisher expansion for the z-distribution

We shall now derive an important expansion by aid of the C.F. expansion of the z-distribution. The C.F. series for a z-variable with f_1 and f_2 degrees of freedom runs as follows (cf. Kendall, 1946, p. 117):

$$P(z \leqq Z; f_1, f_2) = \Phi(\lambda), \tag{4·1}$$

where $$Z = \lambda \sqrt{\left(\frac{\sigma}{2}\right)} - \frac{\delta}{6}(\lambda^2+2) + \sqrt{\left(\frac{\sigma}{2}\right)}\left[\frac{\sigma}{24}(\lambda^3+3\lambda) + \frac{\delta^2}{72\sigma}(\lambda^3+11\lambda)\right] + \ldots, \tag{4·2}$$

and $\Phi(\lambda) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}u^2}\,du$ is the normal distribution function. Here, for brevity, $\sigma = 1/f_1 + 1/f_2$ and $\delta = 1/f_1 - 1/f_2$.

Inserting the values (3·3) of f_1 and f_2 and using the notation (3·4), we find

$$\sigma = \frac{1}{2(n+1)}\frac{1}{p^*q^*}, \quad \delta = \frac{1}{2(n+1)}\frac{q^*-p^*}{p^*q^*}.$$

We insert these expressions in (4·2) and substitute the right member of (3·6) for $2Z$. Combining (3·5) and (4·1) and changing the sign of λ, we finally obtain

$$P(x \geqq a;\, p) = 1 - \Phi(\lambda), \tag{4·3}$$

where a, p and λ satisfy the relation

$$\log p/q - \log p^*/q^* = c_1(n+1)^{-\frac{1}{2}} + c_2(n+1)^{-1} + \ldots \tag{4·4}$$

with $$c_1 = -\lambda(p^*q^*)^{-\frac{1}{2}}, \quad c_2 = \frac{\lambda^2+2}{6}(p^*-q^*)(p^*q^*)^{-1},$$

$$c_3 = -\frac{\lambda^3+5\lambda}{36}(p^*q^*)^{-\frac{1}{2}} + \frac{\lambda^3+11\lambda}{36}(p^*q^*)^{-\frac{1}{2}}, \quad \text{etc.}$$

The formulae (4·3) and (4·4) constitute the starting-point for the investigations in the next section.

5. THE CORNISH-FISHER EXPANSION FOR A TRANSFORMED BINOMIAL VARIABLE

Equations (4·3) and (4·4) contain, as is easily seen, solutions of both the tail problem and the confidence problem in terms of a normal deviate. Before we discuss the solutions of these problems in detail in § 6 and the following sections, we shall apply the C.F. method, not to $\log p/q - \log p^*/q^*$, but to a general function $\phi(p) - \phi(p^*)$. It will later be realized that interesting results are obtained in this way.

We introduce a general transformation $\phi(p)$, where the function $\phi(x)$ is supposed to define a one-to-one correspondence between points in some interval $0 < a \leqq x \leqq b < 1$ on the x-axis and a corresponding interval on the ϕ-axis. Further, $\phi(x)$ is supposed to have bounded derivatives $\phi'(x)$, $\phi''(x)$, ... in the interval. By $p'(\phi)$, $p''(\phi)$, ... we denote the derivatives of the inverse function $p = p(\phi)$ with respect to ϕ (which are also supposed to be bounded). The value $\phi(p^*)$ will sometimes for brevity be denoted by ϕ^*.

We shall replace (4·4) by two other expansions, both of which will be found useful in what follows; the first expansion can be written in the form

$$\phi(p) = \phi(p^*) + a_1(n+1)^{-\frac{1}{2}} + a_2(n+1)^{-1} + a_3(n+1)^{-\frac{3}{2}} + \ldots. \tag{5·1}$$

The coefficients a_ν depend upon λ and p^*, not upon p. We shall call this series the first transformation series.

The second series to be derived in this section is of the same form

$$\phi(p^*) = \phi(p) + b_1(n+1)^{-\frac{1}{2}} + b_2(n+1)^{-1} + b_3(n+1)^{-\frac{3}{2}} + \ldots, \tag{5·2}$$

where the coefficients b_ν only depend upon λ and p, not upon p^*. We call this the second transformation series.

We begin by determining the coefficients a_ν. Expanding the left member of (4·4) formally in a Taylor series around ϕ^*, we find

$$\log p/q - \log p^*/q^* = d_1^*(\phi - \phi^*) + \frac{d_2^*}{2!}(\phi - \phi^*)^2 + \ldots, \tag{5·3}$$

where d_1^*, etc., are the quantities obtained, if p is replaced by p^* in

$$\left.\begin{aligned} d_1 &= \frac{p'}{pq} = \frac{1}{\phi' pq}, \\ d_2 &= \frac{p''}{pq} + \frac{p'^2(p-q)}{p^2q^2} = -\frac{1}{\phi'^2 pq}\left(\frac{\phi''}{\phi'} + \frac{q-p}{pq}\right), \end{aligned}\right\} \tag{5·4}$$

and so on. We determine a_1, a_2, ... successively by inserting the series (5·1) in (5·3) and comparing the coefficients of $(n+1)^{-\frac{1}{2}\nu}$ in the right member of (4·4) and (5·3). We give the first two terms of the resulting series:

$$\left.\begin{aligned} a_1 &= -\lambda(p^*q^*)^{\frac{1}{2}}\phi'(p^*), \\ a_2 &= \tfrac{1}{2}\lambda^2[p^*q^*\phi''(p^*) + (1-k(\lambda))\,(q^*-p^*)\,\phi'(p^*)], \end{aligned}\right\} \tag{5·5}$$

where

$$k(\lambda) = \frac{\lambda^2+2}{3\lambda^2}. \tag{5·6}$$

The coefficients b_ν may be determined by inversion of (5·1) or directly by a modification of the above procedure as follows. We expand $\log p/q - \log p^*/q^*$ around ϕ instead of ϕ^*, which gives

$$2Z = \log p/q - \log p^*/q^* = -d_1(\phi^* - \phi) - \frac{d_2}{2!}(\phi^* - \phi)^2 + \ldots, \tag{5·7}$$

where d_1 and d_2 are given by (5·4). Further, the coefficients c_ν in (4·4) are also expanded around ϕ. After substitution of (5·2) in the resulting series and in (5·7), the coefficients of $(n+1)^{-\frac{1}{2}\nu}$ are compared. We then obtain

$$\left.\begin{aligned} b_1 &= \lambda(pq)^{\frac{1}{2}}\phi'(p), \\ b_2 &= \tfrac{1}{2}\lambda^2[pq\phi''(p) + k(\lambda)\,(q-p)\,\phi'(p)]. \end{aligned}\right\} \tag{5·8}$$

Theoretically, any transformation which gives the two series (5·1) and (5·2) reasonable asymptotic properties may be used for solving the problems treated in this paper.

We finish this section by remarking that (5·1) and (5·2) can be derived directly without using the C.F. expansion of the z-distribution as an intermediary step.

6. Determination of the 'best' transformation of a binomial variable

In many practical situations only the first two terms in (5·1) or (5·2) are calculated, and the terms of order $(n+1)^{-1}$ and higher order are neglected. It is then natural to choose the transformation $\phi(p)$ in such a way that the term of order $(n+1)^{-1}$ vanishes or at least becomes very small. An application of this principle gives of course no guarantee that the total error will be small, but good results may be expected. This is also confirmed by some empirical calculations described later on in the text.

The term of order $(n+1)^{-1}$ is in general a function of the normal deviate λ, and it is thus evident that the desired transformation will contain λ as a parameter. This may seem embarrassing, particularly from the point of view of the tail problem, where our main task consists in calculating λ. With some *a priori* knowledge of the area of the tail we shall, however, be able to overcome this difficulty.

The coefficient a_2 is given in (5·5) and may be written

$$a_2 = \tfrac{1}{2}\lambda^2 p^* q^* \left[\phi''(p^*) + \alpha_1(\lambda)\left(\frac{1}{p^*} - \frac{1}{q^*}\right)\phi'(p^*)\right],$$

where

$$\alpha_1(\lambda) = 1 - k(\lambda) = 1 - \frac{\lambda^2+2}{3\lambda^2}. \tag{6·1}$$

Putting a_2 equal to zero and replacing p^* by p, we obtain the differential equation

$$\frac{\phi''(p)}{\phi'(p)} = \alpha_1(\lambda)\left(\frac{1}{1-p} - \frac{1}{p}\right). \tag{6·2}$$

In terms of a general parameter α, this equation has, apart from a multiplicative constant, the solution

$$\phi(p;\alpha) = \int_{p_0}^{p} x^{-\alpha}(1-x)^{-\alpha}\,dx. \tag{6·3}$$

Starting with b_2 in (5·2) and (5·8) and proceeding analogously, we find the same transformation with the parameter

$$\alpha_2(\lambda) = k(\lambda) = \frac{\lambda^2+2}{3\lambda^2}. \tag{6·4}$$

We observe that

$$\alpha_1(\lambda) + \alpha_2(\lambda) = 1.$$

When λ varies from 0 to $+\infty$, $\alpha_1(\lambda)$ increases from $-\infty$ to $\frac{2}{3}$, and $\alpha_2(\lambda)$ decreases from $+\infty$ to $\frac{1}{3}$. For $\lambda = 2$, we have $\alpha_1 = \alpha_2 = \frac{1}{2}$.

For obvious reasons we shall call (6·3) the beta-transformation. It is interesting to note that most of the transformations used hitherto in the literature are special cases of this transformation. This will become clear in the following sections, where special values will be assigned to α.

The expansions (5·1) and (5·2) can be simplified when the beta-transformation is used. Substituting $\phi'(p^*)$, $\phi''(p^*)$, ... by the expressions obtained by differentiating (6·3), the first transformation series becomes

$$P(x \geqq a;\, p) = 1 - \Phi(\lambda), \tag{6·5}$$

where

$$\phi(p;\alpha) = \phi(p^*;\alpha) - \lambda(p^*q^*)^{\frac{1}{2}-\alpha}(n+1)^{-\frac{1}{2}} + \frac{(2-3\alpha)\lambda^2 - 2}{6}(q^*-p^*)(p^*q^*)^{-\alpha}(n+1)^{-1} + \ldots. \tag{6·6}$$

If we solve with respect to λ in the second term of the right member, we obtain

$$\lambda = (\phi^* - \phi)(p^*q^*)^{\alpha-\frac{1}{2}}(n+1)^{\frac{1}{2}} + \frac{(2-3\alpha)\lambda^2 - 2}{6}(q^*-p^*)(p^*q^*)^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}} + \ldots. \tag{6·7}$$

The second transformation series leads to (6·5) with

$$\lambda = (\phi^* - \phi)(pq)^{\alpha-\frac{1}{2}}(n+1)^{\frac{1}{2}} + \frac{(3\alpha-1)\lambda^2 - 2}{6}(q-p)(pq)^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}} + \ldots. \tag{6·8}$$

The quantities p^* and q^* which appear in the formulae have been defined in (3·4). It is seen that the part of the expansion (6·8) reproduced here is obtained from (6·7) by replacing p^* and q^* by p and q and α in the second term of the right member by $1-\alpha$.

The terms of order $(n+1)^{-1}$ in (6·7) and (6·8) have also been determined and are reproduced here primarily in order to provide a check for formulae published before. In both series $(n+1)^{-1}$ should be multiplied by the factor

$$\frac{\lambda}{36}\left(A-\frac{B}{pq}\right). \tag{6·9}$$

In (6·7) we have

$$\left.\begin{aligned} A &= [1+12(\alpha-1)(2\alpha-1)]\lambda^2+48\alpha-37, \\ B &= [1+6\alpha(\alpha-1)]\lambda^2+12\alpha-7, \end{aligned}\right\} \tag{6·10}$$

and in (6·8)

$$\left.\begin{aligned} A &= [1+12\alpha(2\alpha-1)]\lambda^2-48\alpha+11, \\ B &= (6\alpha^2-\tfrac{1}{2})\lambda^2-12\alpha-1. \end{aligned}\right\} \tag{6·11}$$

We have hitherto only studied the right (upper) tail of the binomial distribution. The other tail is, of course, obtained by substituting $a+1$ for a and taking the complementary probability. Thus

$$P(x \leqq a;\, p) = \Phi(\lambda). \tag{6·12}$$

From (3·4) it follows that p^* and q^* should now be defined by

$$p^* = \frac{a+1}{n+1}, \quad q^* = \frac{n-a}{n+1}.$$

With this modification, the formulae (6·5)–(6·11) are true even in this case.

The reader's attention is drawn to the fact that the equations presented in this section are given in terms of $n+1$, not in terms of n. Also, p^* is defined with $n+1$ in the denominator. In this way a sort of 'automatic' continuity correction is made which replaces the classical half-correction. Freeman & Tukey (1949) have introduced the same continuity correction from heuristic considerations.

7. Application of the beta-transformation to the confidence problem for a binomial variable

It is evident that the first transformation series in the form (6·6) provides the solution of the confidence problem.

Let the desired two-sided confidence level be $1-2\varepsilon$. Let λ_ε be defined by

$$\Phi(\lambda_\varepsilon) = 1-\varepsilon. \tag{7·1}$$

For a given value of α we insert $\lambda = \lambda_\varepsilon$ and $p^* = a/(n+1)$ in the first terms of (6·6) and calculate p from the inverse of ϕ. According to (6·5) and (7·1) the value of p is an approximative solution of $P(x \geqq a;\, p) = \varepsilon$, and is consequently an approximation to the lower confidence limit p_l. With $\lambda = -\lambda_\varepsilon$ and $p^* = (a+1)/(n+1)$ we obtain the upper confidence limit p_u by a similar procedure.

If we take the special value of α obtained by substituting $\alpha = \alpha_1(\lambda_\varepsilon)$ given by (6·1) in (6·6), we have what has been previously called the best transformation. It follows from the results of the preceding section that, in this special case, the term of order $(n+1)^{-1}$ vanishes in (6·6) so that a comparatively simple formula is obtained.

From a practical point of view, the method outlined above is quite unimportant, as (i) excellent tables of confidence limits have been published (Hald, 1952; Mainland, 1948); (ii) tables of percentage points of the incomplete beta and F distributions can be used for

this purpose (cf. § 3). Theoretically the method is, however, of a certain interest, and we shall therefore illustrate the dependence of the best transformation upon the prescribed confidence level by assigning some special values to the normal deviate in (7·1). In the series the term of order $(n+1)^{-1}$ ('the skewness correction') is also given, as it increases the applicability of the expansions to cases where this term differs from zero.

Case (i). $\lambda = 1, \alpha_1(\lambda) = 0$.

When λ is near 1 ($\epsilon \sim 16\,\%$), it is 'best' not to make any transformation at all, and (6·6) becomes

$$p = p^* - \lambda \sqrt{\left(\frac{p^*q^*}{n+1}\right)} + \frac{\lambda^2 - 1}{3} \frac{q^* - p^*}{n+1} + \ldots. \tag{7·2}$$

Case (ii). $\lambda = 2, \alpha_1(\lambda) = \frac{1}{2}$.

When ϵ is near 2·5 %, the inverse sine transformation is best, and the formula (6·6) is specialized to

$$2 \arcsin \sqrt{p} = 2 \arcsin \sqrt{p^*} - \lambda (n+1)^{-\frac{1}{2}} + \frac{\lambda^2 - 4}{12} (q^* - p^*) (p^* q^*)^{-\frac{1}{2}} (n+1)^{-1} + \ldots, \tag{7·3}$$

or, if the angle $\phi^* = \phi(p^*)$ is introduced in the last term,

$$\phi = \phi^* - \lambda (n+1)^{-\frac{1}{2}} + \frac{\lambda^2 - 4}{6} (n+1)^{-1} \cot \phi^* + \ldots.$$

When 95 % two-sided confidence intervals are required, it is thus advantageous to use the inverse sine transformation. This interesting property of the transformation shows that it may serve the double purpose of both stabilizing the variance and furnishing approximations to the tail of the binomial distribution.

Case (iii). An interesting situation arises when λ is large. We should then theoretically (but not practically!) apply the following transformation:

$$\phi(p; \tfrac{3}{4}) = \int_0^p \frac{dx}{x^{\frac{1}{4}}(1-x)^{\frac{1}{4}}}. \tag{7·3'}$$

It is interesting to compare (7·2) and (7·3) with the classical methods used in this situation. Two classical formulae will be mentioned here.

If a binomial variable is expanded directly in a C.F. series, and the expansion is solved with respect to p, the following equations are obtained (half-correction being used):

$$\left.\begin{aligned} p_l &= p^* - \tfrac{1}{2}n - \lambda_\epsilon \sqrt{\left(\frac{p^*q^*}{n}\right)} + \frac{2\lambda_\epsilon^2 + 1}{6n} (q^* - p^*) + \ldots, \\ p_u &= p^* + \tfrac{1}{2}n + \lambda_\epsilon \sqrt{\left(\frac{p^*q^*}{n}\right)} + \frac{2\lambda_\epsilon^2 + 1}{6n} (q^* - p^*) + \ldots, \end{aligned}\right\} \tag{7·4}$$

with $p^* = a/n$ in both formulae.

A related type of formulae is constructed by solving the equation

$$a \pm \tfrac{1}{2} - np = \lambda_\epsilon \sqrt{(npq)} \tag{7·5}$$

with respect to p (cf. Cramér, 1946, p. 515).

In Table 1 a numerical comparison is made between (7·2), (7·3) and (7·4) and the formulae obtained by solving (7·5). The study is very incomplete, but it gives nevertheless some idea of the behaviour of the formulae given in this paper in comparison with those obtained by classical methods. The skewness correction has *not* been included in the calculations, and so the formulae have been used in their simplest form.

Table 1. *Lower confidence limit p_l computed from formulae (7·2)–(7·5) without skewness correction. $n = 49$*

Confidence level (two-sided)	a	p_l (exact)	p_l computed from			
			(7·2)	(7·3)	(7·4)	(7·5)
80 %	5	0·050	0·046	0·052	0·036	0·002
	10	0·131	0·128	0·133	0·120	0·132
	15	0·220	0·217	0·221	0·212	0·220
	20	0·312	0·311	0·313	0·308	0·313
	25	0·410	0·409	0·410	0·408	0·410
95 %	5	0·034	0·017	0·033	0·007	0·001
	10	0·102	0·089	0·102	0·081	0·107
	15	0·182	0·173	0·182	0·167	0·187
	20	0·270	0·264	0·270	0·260	0·273
	25	0·363	0·361	0·363	0·360	0·365
99 %	5	0·023	<0	0·019	<0	<0
	10	0·080	0·054	0·077	0·046	0·089
	15	0·152	0·133	0·150	0·126	0·161
	20	0·234	0·222	0·232	0·217	0·241
	25	0·323	0·318	0·322	0·316	0·327

The value p_l (exact) has been obtained by interpolation in Thompson's (1941) tables.

It is seen from the table that the inverse sine formula gives excellent 95 % values. This confirms the theory developed in the paper. The 80 and 99 % values are also good, better than the values obtained by aid of the other formulae.

The formula obtained by solving (7·5) can be ranked as second. It is, however, remarkably inaccurate for $a = 5$.

(7·2) is rather inaccurate except in the 80 % case, where it is fairly good (in accordance with the theory).

The classical formula (7·4) provides poor values. If it is used, the skewness correction (last term in the right member) ought to be included.

The numerical investigations need to be extended but indicate that: (i) the inverse sine method is better than the classical methods for computing confidence limits, if skewness corrections are not used; (ii) the expression obtained by solving (7·5) should not be used for computational purposes as it is complicated and rather inaccurate. It is better to replace (7·5) by the inverse sine formula or, if tables of the inverse sine transformation should not be available, by (7·4) with skewness correction.

8. Application of the beta-transformation to the binomial tail problem

We shall continue our study of the beta-transformation by applying it to the tail problem. As it is rather obvious how to proceed, we shall only make a few remarks on this question without entering upon any discussion of the relative merits of the different formulae.

The problem consists in determining the normal deviate λ which 'corresponds' to the tail of the binomial distribution. When λ has been calculated, we immediately obtain $P(x \geqq a; p)$ from (6·5) or $P(x \leqq a; p)$ from (6·12). The determination of λ can be made in a variety of ways with the tools now at our disposal.

First, we have to choose between (6·7) and (6·8). (Formulae which do not contain λ in the right member may easily be constructed, but we prefer to use the formulae as they stand.) We shall only discuss (6·8), as the application of (6·7) is similar.

We shall assume that only the first term in the right member of (6·8) is computed (if higher accuracy is required the computations can be performed in two stages, the second term being included in the last stage). Provided that we know something in advance about the magnitude of λ, the equation (6·4) should, theoretically, govern our choice of α (we disregard here altogether the obvious fact that it is more convenient to use one single transformation and include a sufficient number of terms of the asymptotic series). If the same values are assigned to λ as in the last section, we have the following transformations:

Case (i). $\lambda = 1$, $\alpha_2(\lambda) = 1$.

$$\phi(p;\, 1) = \log p/q - \log p_0/q_0. \tag{8·1}$$

The transformation series is specialized to a formula which is very similar to the origin (4·4) of all our subsequent investigations. Apart from an obvious rearrangement, the two expansions differ only in so far as the roles of p and p^* are interchanged. Thus we find that the logarithmic function which connects the binomial distribution with the z-distribution is a prominent member of the class of beta-transformations.

The logarithmic transformation has previously been used by Dyke & Patterson (1952) in an analysis of variance of attributive data.

Case (ii). $\lambda = 2$, $\alpha_2(\lambda) = \frac{1}{2}$.

Precisely as when the first transformation series is used, the inverse sine transformation is 'best'. The first few terms of the expansion (6·8) run in this case as follows:

$$\lambda = (2 \arcsin \sqrt{p^*} - 2 \arcsin \sqrt{p})\,(n+1)^{\frac{1}{2}} + \frac{\lambda^2 - 4}{12}(q-p)\,(pq)^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}} + \ldots. \tag{8·2}$$

This formula should be compared with a very similar expression which Freeman & Tukey (1949, 1950) have constructed empirically. In their formula, our quantity $\dfrac{q-p}{\sqrt{\{(n+1)\,pq\}}}$ is replaced by $\dfrac{1}{\sqrt{(np+1)}} - \dfrac{1}{\sqrt{(nq+1)}}$.

Case (iii). $\lambda \to \infty$, $\alpha_2(\lambda) = \frac{1}{3}$.

This leads to the transformation

$$\phi(p;\, \tfrac{1}{3}) = \int_0^p \frac{dx}{x^{\frac{1}{3}}(1-x)^{\frac{1}{3}}}. \tag{8·3}$$

No numerical comparison of the formulae applicable to the binomial tail problem has been included in this paper. The reader is referred to § 7, where the behaviour of (8·2) without skewness correction has been studied, and to § 9, where the solution of the Poisson tail problem has been illustrated by some numerical examples. The results of the numerical investigation reported in § 7 indicate that the inverse sine method can be recommended for practical use, both when confidence limits are required and when the tail of the binomial distribution is to be estimated.

9. Transformations of the Poisson and χ^2 distributions

(a) *Derivation of Cornish-Fisher expansions for a transformed Poisson variable. Determination of the best transformation*

The transformation problem for a Poisson variable can be treated in a similar way to the corresponding problem for a binomial variable. For brevity, we shall here only state the main results.

Let the Poisson variable x have the mean m. We put

$$P(x \geqq a;\, m) = 1 - \Phi(\lambda). \tag{9·1}$$

We introduce a general transformation $\phi(x)$ and derive expressions of the same kind as (5·1) and (5·2). The first transformation series is found to be

$$\phi(m) = \phi(a) - \lambda\phi'(a)\, a^{\frac{1}{2}} + \tfrac{1}{2}\lambda^2[a\phi''(a) + \alpha_1(\lambda)\,\phi'(a)] + \ldots, \tag{9·2}$$

where $\alpha_1(\lambda)$ is given by (6·1). Equating the expression within brackets to zero, replacing $\alpha_1(\lambda)$ by α and a by m, we have a differential equation which has the solution

$$\phi(m) = \int_{m_0}^{m} x^{-\alpha}\, dx = \begin{cases} \dfrac{1}{1-\alpha}\, m^{1-\alpha} + \text{const.} & (\alpha < 1), \\ \log m/m_0 & (\alpha = 1). \end{cases} \tag{9·3}$$

We then find the following series:

$$\phi(m) = \phi(a) - \lambda a^{\frac{1}{2}-\alpha} + \frac{(2-3\alpha)\,\lambda^2 - 2}{6}\, a^{-\alpha} - \tfrac{1}{36}a^{-\frac{1}{2}-\alpha}[(1 + 6\alpha(\alpha-1))\,\lambda^3 + (12\alpha - 7)\,\lambda] + \ldots. \tag{9·4}$$

(9·3) and (9·4) may be considered as limiting expressions of (6·3) and (6·6). We easily realize this by multiplying both members of (6·6) by $(n+1)^{1-\alpha}$ and letting n tend to infinity. Passing to the limit in (6·8), we obtain analogously the second transformation series

$$\lambda = [\phi(a) - \phi(m)]\, m^{\alpha-\frac{1}{2}} + \frac{(3\alpha-1)\,\lambda^2 - 2}{6}\, m^{-\frac{1}{2}} - \tfrac{1}{72}[(12\alpha^2 - 1)\,\lambda^3 - (24\alpha + 2)\,\lambda]\, m^{-1} + \ldots. \tag{9·5}$$

The discussion of the choice of α is made exactly as in the binomial case. The results are summarized below.

(b) *Determination of confidence limits for the mean of the Poisson distribution*

From (9·4) we obtain three confidence formulae by giving λ the same values as in the binomial case, viz. 1, 2 and ∞. We then find

Case (i). $\lambda = 1$, $\alpha_1(\lambda) = 0$.

$$m = a - \lambda a^{\frac{1}{2}} + \frac{\lambda^2 - 1}{3} - \frac{\lambda^3 - 7\lambda}{36}\, a^{-\frac{1}{2}} + \ldots. \tag{9·6}$$

Case (ii). $\lambda = 2$, $\alpha_1(\lambda) = \frac{1}{2}$.

$$m^{\frac{1}{2}} = a^{\frac{1}{2}} - \lambda + \frac{\lambda^2 - 4}{12}\, a^{-\frac{1}{2}} + \frac{\lambda^3 + 2\lambda}{72}\, a^{-1} + \ldots. \tag{9·7}$$

Case (iii). $\lambda \to \infty$, $\alpha_1(\lambda) = \frac{2}{3}$.

$$m^{\frac{1}{2}} = a^{\frac{1}{2}}\left[1 - \frac{\lambda}{3}\, a^{-\frac{1}{2}} - \tfrac{1}{9}a^{-1} + \frac{\lambda^3 - 3\lambda}{324}\, a^{-\frac{3}{2}} + \ldots\right]. \tag{9·8}$$

The series (9·6) is known (Campbell, 1923; Riordan, 1949). The cube-root transformation used in (9·8) has previously been suggested by Haldane (1948). It can be considered as a limiting case of (7·3′).

The lower confidence limit m_l may be obtained from any of these formulae by substituting $\lambda = \lambda_\varepsilon$, and the upper limit m_u by substituting $\lambda = -\lambda_\varepsilon$ and replacing a by $a+1$.

(c) *Determination of percentage points of the χ^2-distribution*

The last three formulae can also be used for calculating percentage points of the χ^2-distribution. We then use the well-known relation (cf. (3·2))

$$P(x \geqq a;\, m) = P(\chi^2_{2a} \leqq 2m). \tag{9·9}$$

Consequently, in view of (9·1), we have

$$P(\chi^2_{2a} \leqq 2m) = 1 - \Phi(\lambda). \tag{9·10}$$

Let us denote the degrees of freedom of the χ^2-variable by f. Replacing a by $\frac{1}{2}f$, m by $\frac{1}{2}\chi^2$ and λ by $-\lambda$ in (9·10) and (9·4) we find for $\alpha < 1$

$$\frac{1}{1-\alpha}(\chi^2)^{1-\alpha} = \frac{1}{1-\alpha}f^{1-\alpha} + \lambda \cdot 2^{\frac{1}{2}} f^{\frac{1}{2}-\alpha} + \frac{(2-3\alpha)\lambda^2 - 2}{3} f^{-\alpha}$$
$$+ \tfrac{1}{18} \cdot 2^{\frac{1}{2}}[(1 + 6\alpha(\alpha-1))\lambda^3 + (12\alpha - 7)\lambda] f^{-\frac{1}{2}-\alpha} + \ldots. \tag{9·11}$$

For $\alpha = 0$ we have a series given by Peiser (1943) and Goldberg & Levine (1946). The value $\alpha = \frac{1}{2}$ leads to

$$\sqrt{(2\chi^2)} = \sqrt{(2f)} + \lambda + \frac{\lambda^2 - 4}{6}\frac{1}{\sqrt{(2f)}} - \frac{\lambda^3 + 2\lambda}{18}\frac{1}{2f} + \ldots. \tag{9·12}$$

It is of interest to compare this formula with Fisher's χ^2-approximation $\sqrt{(2\chi^2)} = \sqrt{(2f-1)} + \lambda$. Finally, for $\alpha = \frac{2}{3}$ we obtain after dividing by $3f^{\frac{1}{2}}$ and putting $c = 2/(9f)$

$$\left(\frac{\chi^2}{f}\right)^{\frac{1}{2}} = 1 + \lambda c^{\frac{1}{2}} - c - \frac{\lambda^3 - 3\lambda}{12} c^{\frac{1}{2}} + \ldots. \tag{9·13}$$

If the last term is neglected we have Wilson & Hilferty's (1931) χ^2-approximation. Thus we conclude that the results of these authors may be regarded as variations of one single theme. The three formulae are all of them accurate, if several terms are computed, but if only the two first terms are retained at the calculations, the desired confidence level should determine which formula is to be preferred.

We finally observe that the logarithmic transformation obtained for $\alpha = 1$ has been applied to the χ^2-distribution by Bartlett & Kendall (1946).

(d) *Determination of the tail of the Poisson distribution*

The area of a tail of the Poisson distribution can, in analogy with the binomial case, be calculated either from (9·4) or from (9·5) by assigning special values to α. Similarly, in view of (9·9), we can determine by this means an approximation to the tail integral of the χ^2-distribution.

We shall list a few of these formulae in a form suitable for computation. If we omit the last term in (9·8) and solve with respect to λ, we have

$$\lambda = 3\sqrt{a}\left(1 - \frac{\sqrt[3]{m}}{\sqrt[3]{a}}\right) - \frac{1}{3\sqrt{a}}. \tag{9·14}$$

If we put $\alpha = \frac{1}{3}$ in (9·5) we get a similar formula

$$\lambda = \tfrac{3}{2}\sqrt{m}\left[\left(\frac{\sqrt[3]{a}}{\sqrt[3]{m}}\right)^2 - 1\right] - \frac{1}{3\sqrt{m}}. \qquad (9{\cdot}15)$$

The transformation $a^{\frac{1}{3}}$ used in this equation can be considered as a limiting case of (8·3).

Table 2. *Numerical study of the accuracy of formulae* (9·14)–(9·17)

Mean	a	λ(exact)	λ computed from			
			(9·14)	(9·15)	(9·16)	(9·17)
$m=1$	1	−0·34	−0·33	−0·33	0·00	−0·33
	2	0·63	0·64	0·55	0·83	0·55
	3	1·40	1·40	1·29	1·46	1·31
	4	2·08	2·05	1·95	2·00	2·00
	5	2·68	2·64	2·55	2·47	2·65
	6	3·24	3·17	3·12	2·90	3·27
	7	3·77	3·66	3·66	3·29	3·86
	8	4·27	4·12	4·17	3·66	4·44
$m=10$	1	−3·92	−3·80	−3·83	−4·32	−3·94
	2	−3·29	−3·25	−3·23	−3·50	−3·28
	3	−2·77	−2·76	−2·72	−2·86	−2·75
	4	−2·31	−2·31	−2·27	−2·32	−2·29
	5	−1·89	−1·89	−1·86	−1·85	−1·87
	10	−0·11	−0·11	−0·11	0·00	−0·11
	15	1·38	1·38	1·37	1·42	1·37
	16	1·66	1·66	1·64	1·68	1·64
	17	1·93	1·92	1·91	1·92	1·91
	18	2·19	2·19	2·17	2·18	2·18
	20	2·70	2·69	2·68	2·62	2·70
	25	3·90	3·88	3·89	3·68	3·93
	26	4·13	4·11	4·12	3·87	4·16
	27	4·36	4·33	4·35	4·07	4·40

In Table 2 a numerical comparison is made between (9·14), (9·15) and the formula obtained by taking $\alpha = \frac{1}{2}$ in (9·5). This formula may be used in two ways:

(*a*) $$\lambda = 2\sqrt{a} - 2\sqrt{m}; \qquad (9{\cdot}16)$$

(*b*) $\lambda = \lambda_0$ is computed from (9·16), and the value then obtained is inserted in

$$\lambda = \lambda_0 + \frac{\lambda_0^2 - 4}{12}\frac{1}{\sqrt{m}}. \qquad (9{\cdot}17)$$

In the table, the column denoted by λ(exact) gives the correct value of λ computed by means of Molina's (1947) table from (9·1). Apart from the fact that (9·16) gives poor values when λ is very different from the value $\lambda = 2$, all the formulae are remarkably accurate. It may be remarked that Freeman & Tukey, in their empirical study of different approximation formulae, have used (9·17) in a slightly modified version.

10. TRANSFORMATIONS OF THE NEGATIVE BINOMIAL DISTRIBUTION

The theory which has been developed in the preceding sections can without difficulty be extended to the case of a negative binomial variable. Only some results will be briefly mentioned here.

We denote the parameters of the negative binomial variable x by N, P and $Q = 1+P$. The probability that x assumes a given value A is given by

$$P(x=A) = \binom{N+A-1}{A} P^A Q^{-N-A}.$$

We introduce the notation $P^* = A/N$. The analogues of (5·1) and (5·2) are of the form

$$\psi(P) = \psi(P^*) + A_1 N^{-\frac{1}{2}} + A_2 N^{-1} + \ldots, \tag{10·1}$$

where the coefficients A_ν depend upon P^*, but not upon P, and

$$\psi(P) = \psi(P^*) + B_1 N^{-\frac{1}{2}} + B_2 N^{-1} + \ldots, \tag{10·2}$$

where the coefficients B_ν depend upon P, not upon P^*.

The coefficient of N^{-1} in both expansions vanishes, if $\psi(P)$ is of the form

$$\psi(P; \alpha, \beta) = \int_{P_0}^{P} x^{-\alpha}(1+x)^{-\beta}\, dx. \tag{10·3}$$

In (10·1) the parameters α and β should be chosen as follows:

$$\alpha = 1 - \frac{\lambda^2+2}{3\lambda^2}, \quad \beta = \frac{2(\lambda^2+2)}{3\lambda^2}. \tag{10·4}$$

In (10·2) the parameter values are

$$\alpha = \frac{\lambda^2+2}{3\lambda^2}, \quad \beta = 1 - \frac{2(\lambda^2+2)}{3\lambda^2}. \tag{10·5}$$

By assigning special values to λ, a variety of different transformations is obtained. We shall not enter upon any complete discussion of these questions and only make two short remarks.

(i) If we take $\lambda = 2$ in (10·5), we find $\alpha = \frac{1}{2}$, $\beta = 0$ and $\psi(P) = 2P^{\frac{1}{2}}$. Thus, when the tail of the negative binomial distribution corresponds to a normal deviate equal to about 2, a simple square root transformation is 'best', not an inverse sinh transformation, which the solution of the corresponding problem in the binomial case (cf. § 8) might possibly suggest to those led by intuition!

(ii) The inverse sinh transformation is not 'best' for any value of λ in the meaning which we have assigned to the term in this paper. It belongs, however, to the class of functions (10·3) and is obtained for $\alpha = \beta = \frac{1}{2}$.

We finally remark that the formulae which are relevant in the negative binomial case can be derived either from first principles by means of the relationship between this distribution and the beta-distribution or by translating the formulae obtained in the binomial case in an appropriate manner.

I am indebted to Prof. E. S. Pearson and to the referees for many helpful suggestions.

REFERENCES

BARTLETT, M. S. & KENDALL, M. G. (1946). *J. R. Statist. Soc. Suppl.* **8**, 128.
CAMPBELL, G. A. (1923). *Bell Syst. Tech. J.* **2**, 95.
CORNISH, E. A. & FISHER, R. A. (1937). *Rev. Inst. Int. Statist.* **4**, 307. (Reprinted in: FISHER, R. A. (1950). *Contributions to Mathematical Statistics.* New York: Wiley and Sons.)
CRAMÉR, H. (1946). *Mathematical Methods of Statistics.* Princeton University Press.
DYKE, G. V. & PATTERSON, H. D. (1952). *Biometrics*, **8**, 1.
EISENHART, C. (1947). *Selected Techniques of Statistical Analysis*, p. 395. New York: McGraw-Hill.
FREEMAN, M. F. & TUKEY, J. W. (1949). *Transformations related to the angular and the square root.* MR-24, Statistical Research Group, Princeton.
FREEMAN, M. F. & TUKEY, J. W. (1950). *Ann. Math. Statist.* **21**, 607.
GOLDBERG, H. & LEVINE, H. (1946). *Ann. Math. Statist.* **17**, 216.
HALD, A. (1952). *Statistical Theory with Engineering Applications. Statistical Tables and Formulas.* New York: Wiley and Sons.
HALDANE, J. B. S. (1948). *Biometrika*, **35**, 297.
KENDALL, M. G. (1946). *The Advanced Theory of Statistics*, **2**. London: Griffin and Co.
MAINLAND, D. (1948). *Canad. J. Res.* E, **26**, 1.
MOLINA, E. C. (1947). *Poisson's Exponential Binomial Limit.* New York: van Nostrand Co.
MOSTELLER, F. & TUKEY, J. W. (1949). *J. Amer. Statist. Ass.* **44**, 174.
PAULSON, E. (1947). *Selected Techniques of Statistical Analysis*, p. 247. New York: McGraw-Hill.
PEISER, A. M. (1943). *Ann. Math. Statist.* **14**, 56.
RIORDAN, J. (1949). *Ann. Math. Statist.* **20**, 417.
THOMPSON, C. M. (1941). *Biometrika*, **32**, 151.
WILSON, E. B. & HILFERTY, M. M. (1931). *Proc. Nat. Acad. Sci. Wash.* **17**, 694.

ADDITIONAL REFERENCES BEARING ON THE SUBJECT NOT MENTIONED IN THE TEXT

ANSCOMBE, F. J. (1948). *Biometrika*, **35**, 246.
ANSCOMBE, F. J. (1950). *J. R. Statist. Soc.* A, **113**, 228.
BAKER, A. G., HILLIER, K. W. & WOODWARD, R. H. (1954). *Appl. Statist.* **3**, 12.
CLARINGBOLD, P. J., BIGGERS, J. D. & EMMENS, C. W. (1953). *Biometrics*, **9**, 467.
CURTISS, J. H. (1943). *Ann. Math. Statist.* **14**, 107.

A QUICKLY CONVERGENT EXPANSION FOR CUMULATIVE HYPERGEOMETRIC PROBABILITIES, DIRECT AND INVERSE

By M. E. WISE

Philips Research Laboratories, N. V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands

An expansion is derived for the cumulative probability P, that is, the sum of the first p terms of a series of hypergeometric probabilities. If a batch of N has a proportion defective x, P can be seen as the probability that less than p defectives will be found in a sample of n. The expansion is in descending powers of *squares* of $N-\frac{1}{2}n+\frac{1}{2}$, as is the series for the inverse, i.e. P given, x unknown; the first two terms of both are obtained. Each is illustrated by a practical example in which several values obtained from the first term alone and from the sum of the first two terms are compared with exact values; the approximations seem more than sufficient for sampling problems except for probabilities extremely near 0 or 1, and are also simple to calculate.

Introduction

In most sampling probability problems we assume, if we can, that the batch sampled is infinite. When this is so, the probability distribution of the value or type of the nth unit drawn from the batch does not depend upon the values of the previous $n-1$ units which have been taken away. We may want to know the probability that not more than c defectives are found in a sample of n when the batch has a proportion x of defective units. Then, if the sampled units are not returned to the batch, this probability P is the sum of the first $c+1$ terms of a hypergeometric series.

No simple way of calculating a hypergeometric series has so far been found. When the sample size n is small, calculating and summing individual terms is not laborious (see, for example, Finney, 1948, Yates, 1934), but it is so for larger n; the inverse problem, given the probability P, to find the corresponding value of x is just as important in practice and looks more troublesome still. Any reasonably accurate approximation seems so complex (see, for example, Davies, 1933), that the correction for a finite batch size is avoided whenever possible. Simpler approximations, for example, those based on a Gaussian distribution, are not too accurate as Katz (1953) has pointed out. However, approximations have now been found that are both simple and accurate, both for an unknown probability given x and for an unknown proportion-defective x given P. The latter expression, surprisingly, is in some ways the easier of the two.

The mathematical treatment is an unusual one in theoretical statistics (though it has no doubt been used in statistical mechanics). For this reason it is given in full and also because it may be useful in some of the many other mathematical problems involving hypergeometric series. For applying the formulae to numerical probability calculations, however, the reader needs only to study the last section (8).

1. Mathematical expression for the probability of obtaining less than p defectives in a sample from a finite batch

This well-known result has to be expressed in a certain mathematical form that is best deduced from first principles. To start with, we derive the probability of obtaining exactly p black balls (or defectives, say) in our sample of n from a batch containing Nx black balls and $N(1-x)$ white ones. We have, of course, $p \leqslant Nx$. The number of ways of choosing p

balls out of Nx numbered balls is the coefficient of t^p in $(1+t)^{Nx}$. The rest of the sample consists of $n-p$ white balls chosen from $N(1-x)$, and the number of ways of choosing them is the coefficient of u^{n-p} in $(1-u)^{N-Nx}$. So the number of possible samples of n that contain p black balls is the coefficient of $t^p u^{n-p}$ in

$$(1+t)^{Nx}(1+u)^{N-Nx}. \tag{1·1}$$

Put $t = uv$, then

$$t^p u^{n-p} = v^p u^n. \tag{1·2}$$

Then the coefficient of u^n in

$$(1+uv)^{Nx}(1+u)^{N-Nx} \tag{1·3}$$

is a power series in v, in which the coefficient of v^p is the number of ways of obtaining exactly p black balls. We want the corresponding probability. The number of possible samples of n is obtained by putting $v = 1$; obviously it is the coefficient of u^n in (1·3), equal to $N!/\{(N-n)!\,n!\}$. Dividing by this gives the generating function for the probability $P_{n,p}$ of obtaining exactly p in a sample of n, as the coefficient of $v^p u^n$, i.e.

$$G_n(v) = \sum_{p=0}^{n} P_{n,p} v^p,$$

and in closed form

$$\frac{N!}{(N-n)!\,n!} G_n(v) = \frac{1}{2\pi i}\int_0 (1+uv)^{Nx}(1+u)^{N-Nx}\frac{du}{u^{n+1}}, \tag{1·4}$$

where the contour of integration 0 goes anti-clockwise round the origin. Let us suppose it does not go round $u = -1$ or $u = -1/v$. Then (1·4) becomes valid for fractional values of Nx, which besides being more satisfying mathematically is particularly useful when dealing with the inverse function. So from now on x may take any value between 0 and 1.

It is worth mentioning some well-known results derivable from (1·4); the hypergeometric distribution is only one of many that are most easily handled by their generating functions. First the well-known probability $P_{n,p}$ is obtained by simply picking out the coefficient of $v^p u^n$ in (1·3):

$$P_{n,p} = \frac{n!(N-n)!\,(Nx)!\,(N-Nx)!}{N!\,p!\,(n-p)!\,(Nx-p)!\,(N-Nx-n+p)!}. \tag{1·5}$$

Hence

$$P_{n,0} = \frac{(N-n)!\,(N-Nx)!}{N!\,(N-Nx-n)!},$$

and the sum of the first p terms is

$$\sum_{r=0}^{p-1} P_{n,r} = P_{n,0}\left\{1 + \frac{nNx}{1!\,(N-Nx-n+1)} + \frac{n(n-1)\,Nx(Nx-1)}{2!\,(N-Nx-n+1)(N-Nx-n+2)} + \ldots\right\}, \tag{1·6}$$

which, in the usual mathematical notation, is the sum of the first p terms of

$$P_{n,0}F\{-n, -Nx, N(1-x)-n+1, 1\},$$

as pointed out by Karl Pearson (1934). This sum ought to be called an 'incomplete hypergeometric function' by analogy with the incomplete beta function.

Further, regarding r as a variate, its moments, etc., are easily obtained by replacing v by $1+v$ in (1·4); whence

$$\frac{N!}{(N-n)!\,n!} G_n(1+v) = \frac{1}{2\pi i}\int_0 (1+u)^N\left(1+\frac{uv}{1+u}\right)^{Nx}\frac{du}{u^{n+1}}, \tag{1·7}$$

in which the coefficients of powers of v in $G_n(1+v)$ now give the factorial moments of r; differentiating $G_n(1+v)$ with respect to v and then putting $v = 0$ gives from the residue at $u = 0$

$$\Sigma r P_r = \bar{r} = nNx/N = nx, \tag{1·8}$$

$$\Sigma r(r-1)P_r = \bar{r^2} - \bar{r} = \frac{n(n-1)\,Nx(Nx-1)}{N(N-1)}, \tag{1·9}$$

and so on to higher factorial moments. From (1·8) and (1·9) the well-known variance of r is easily found:

$$\bar{r^2} - \bar{r}^2 = nx(1-x)\frac{(N-n)}{N-1}.$$

2. A CLOSED EXPRESSION FOR THE PROBABILITY OF OBTAINING LESS THAN p BLACK BALLS

Again the generating function is used. Write

$$\frac{N!}{(N-n)!\,n!}G_n(v) = \binom{N}{n}G_n(v) = \frac{1}{2\pi i}\int_0 (1+u)^N\left(\frac{1+uv}{1+u}\right)^{Nx}\frac{du}{u^{n+1}}. \tag{2·1}$$

This suggests putting $(1+uv)/(1+u) = 1+z$, then when v and u both approach zero, $z/u \to -1$, so the z path is the same as the u path. A few lines of algebra lead to

$$\binom{N}{n}G_n(v) = \frac{(v-1)^{N+1}}{2\pi i}\int_0 \frac{(1+z)^{Nx}(v-1-z)^{n-1-N}dz}{z^{n+1}}. \tag{2·2}$$

The cumulative probability, i.e. the sum of the probabilities of obtaining $0, 1, 2, \ldots$ or $p-1$, is given by

$$\binom{N}{n}\sum_{r=0}^{p-1}P_{n,r} = \binom{N}{n}\frac{1}{2\pi i}\int_{O_v} G_n(v)\left(\frac{1}{v}+\frac{1}{v^2}+\ldots+\frac{1}{v^p}\right)dv\ldots, \tag{2·3}$$

where O_v is a small anti-clockwise path round the origin. Substituting in (2·2) gives

$$\binom{N}{n}\sum_{r=0}^{p-1}P_{n,r} = \left(\frac{1}{2\pi i}\right)^2\int_{O_v}\left\{\int_{O_z}\frac{(1+z)^{Nx}}{z^{n+1}}(v-1-z)^{n-1-N}(v-1)^N(v^{-p}-1)\,dz\right\}dv. \tag{2·4}$$

Let us integrate first with respect to v. Then the path obviously does not enclose the pole at $v = 1+z$, and the only other pole in the v plane is (the multiple one) at the origin. So the factor $v^{-p}-1$ can be replaced by v^{-p}. Then the cumulative probability, called simply P from now on, is given formally by

$$\binom{N}{n}\sum_{r=0}^{p-1}P_{n,r} = \binom{N}{n}P = -\frac{1}{2\pi i}\int_{O_z}\frac{(1+z)^{Nx}F(z)\,dz}{z^{n+1}}, \tag{2·5}$$

where

$$F(z) = \frac{1}{2\pi i}\int_{O_v}\left(\frac{v-1}{v-1-z}\right)^N(v-1-z)^{n-1}\frac{dv}{v^p}. \tag{2·6}$$

3. THE MATHEMATICAL FORM FOR THE 'BEST' EXPANSION

The best expansion, the one for which one or two terms give the best approximation, is *not* in powers of N^{-1}, which is what we would expect; had this been so it would probably have been discovered long ago. But it was not clear at first even what the form of the expansion ought to be.* A guide to what to do is the procedure which succeeded so well with the incomplete beta function and its inverse (Wise, 1950) long after it had been asserted many times, that there was no one expansion good enough over all values of the three variables, and that the inverse was even more difficult to handle.

Beginning with $F(z)$ and remembering that $N > n-1$, since obviously N need not be less than $2(n-1)$, we first transform $(v-1)/(v-1-z)$ into an exponential. To make the new path of integration go round the origin, put

$$(v-1)/(v-1-z) = e^{-\eta}/(1+z). \tag{3·1}$$

* After having found it I feel the first term might have been guessed at, but in fact I only reached it after several transformations of the contour integrals.

Then, when v and z tend to zero, $\eta/(vz)$ tends to 1. But both the v and the z paths can be chosen very close to the origin, and then the η path is still nearer zero and is always inside the z path. This becomes important later on.

We find that

$$F(z) = (-1)^{n-1}\left[\frac{z^n}{2\pi i}(1+z)^{n-p-N}\int_{O_\eta} e^{\eta(n-N)}\{e^\eta(1+z)-1\}^{p-n-1}(e^\eta-1)^{-p}\right]d\eta. \quad (3{\cdot}2)$$

Next we convert the exponentials that do not contain an index N into hyperbolic sines, i.e. write

$$e^\eta(1+z)-1 \equiv e^{\frac{1}{2}\eta}(1+z)^{\frac{1}{2}}\,2\sinh\tfrac{1}{2}\{\eta+\log(1+z)\}. \quad (3{\cdot}3)$$

Substituting in (2·5) leads to

$$\binom{N}{n}P = \{(-1)^n/2\pi i\}\int_{O_z}(1+z)^{-L}F_1(z)\,dz/z, \quad (3{\cdot}4)$$

where $L = N(1-x)-\frac{1}{2}n+\frac{1}{2}p+\frac{1}{2}$ and

$$F_1(z) = \frac{1}{2\pi i}\int_{O_\eta}\{\exp(\tfrac{1}{2}n-N-\tfrac{1}{2})\,\eta\}[2\sinh\tfrac{1}{2}\{\eta+\log(1+z)\}]^{p-n-1}(2\sinh\tfrac{1}{2}\eta)^{-p}\,d\eta. \quad (3{\cdot}5)$$

It is now clear that there is a series expansion for P in descending powers of *squares* of $M = N-\frac{1}{2}n+\frac{1}{2}$. So we put $\eta = -w/M$, and $\log(1+z) = \theta/M$. The paths of integration are again round the origin, but with $|\theta| \gg |w|$ (see equation (3·1) and the paragraph below it). We find that

$$F_1(z) = (-1)^{p+1}/(2\pi iM)\int_{O_w} e^w\left(2\sinh\frac{\theta-w}{2M}\right)^{p-n-1}\left(2\sinh\frac{w}{2M}\right)^{-p}dw \quad (3{\cdot}6)$$

and

$$\binom{N}{n}P = \frac{(-1)^n}{2\pi iM}\int_{O_\theta}\frac{e^{-h\theta}F_1(z)\,d\theta}{2\sinh(\frac{1}{2}\theta/M)}, \quad (3{\cdot}7)$$

where

$$h = (L-\tfrac{1}{2})/M = \{N(1-x)-\tfrac{1}{2}n+\tfrac{1}{2}p\}/(N-\tfrac{1}{2}n+\tfrac{1}{2}). \quad (3{\cdot}8)$$

The sum of the first $n+1$ terms of the series of hypergeometric probabilities—that is, the value of P when $p = n+1$—must be exactly 1. This shows that the factor $\binom{N}{n}$ also has an expansion in powers of M^{-2}. Further, $M = N-\frac{1}{2}n+\frac{1}{2}$ is simply the mean number of unsampled units. This is N at first, then one unit is removed and there are $N-1$ left to choose from, then $N-2$ and so on down to $N-n+1$.

Further, the proportion defective x occurs only in the index $-h\theta$, and as n/N tends to zero h tends to $1-x$. Thus h looks like an effective proportion of 'good' units, and we shall see later that this is just what it is.

So far the calculations are not laborious, but rather tricky, so that we have given intermediate steps. It only remains to expand the function and its inverse. This is straightforward; there is perhaps a shorter way of obtaining the inverse.

4. The first two terms of the expansion for the probability

It is easily found that

$$\{2\sinh(\tfrac{1}{2}w/M)\}^p = (w/M)^p\{1+pw^2/(24M^2)-\ldots\}, \quad (4{\cdot}1)$$

and from this and similar expansions

$$M^{-n}\binom{N}{n}P = \frac{(-1)^{n+p}}{4\pi^2}\int_{O_\theta}\int_{O_w}\frac{e^{-h\theta+w}(\theta-w)^{p-n-1}}{\theta w^p} \times\left\{1-\frac{(n-p+2)(\theta-w)^2+2w(\theta-w)+(p+1)w^2}{24M^2}\right\}dw\,d\theta, \quad (4{\cdot}2)$$

with $|\theta| \gg |w|$. So the first approximation is the double contour integral on the left. We write it as

$$H_h(n-p+1,p) = \frac{(-1)^{n-p+1}}{(2\pi i)^2} \int_{O_\theta} \int_{O_w} \frac{e^{-h\theta+w}\, dw\, d\theta}{\theta(\theta-w)^{n-p+1} w^p}\, dw\, d\theta. \tag{4·3}$$

Then each of the three terms in the coefficient of M^{-2} is a similar H integral.

Now introducing the expansion of the left-hand side (of course it can be found as a special case of the one on the right, but was obtained more easily from simpler contour integrals like those given in § 2), this is*

$$M^{-n}\binom{N}{n} = \frac{1}{n!}\left\{1 - \frac{(n-1)\,n(n+1)}{24M^2} + \text{coefficient of } M^{-4}\right\}, \tag{4·4}$$

whence

$$\frac{P}{n!}\left\{1 - \frac{(n-1)\,n(n+1)}{24M^2}\right\}$$

$$= H_h(n-p+1,p) - \frac{(n-p+2)\,H_h(n-p-1,p) - 2H_h(n-p,p-1) + (p+1)\,H_h(n-p+1,p-2)}{24M^2}$$

$$+ \text{term in } M^{-4}, \tag{4·5}$$

and it only remains to find the H_h integral. By putting $\theta = w+t$ and integrating once, the resulting integral was recognized as an incomplete beta function, which was to be hoped for, for if $M \to \infty$ but n stays constant, we should obtain a sum of binomial probabilities in the limit. This would prove that H is an incomplete beta function, but it is a useful check to verify this directly, by differentiating with respect to h:

$$\frac{\partial H}{\partial h} = \frac{(-1)^{n-p}}{(2\pi i)^2} \int_{O_\theta} \int_{O_w} \frac{e^{-h\theta+w}\, dw\, d\theta}{(\theta-w)^{n-p+1} w^p} \quad (|\theta| \gg |w|). \tag{4·6}$$

Put $\theta = w+t$; the path in the t plane must go round $t = 0$. Then

$$\frac{\partial H}{\partial h} = \frac{(-1)^{n-p}}{(2\pi i)^2} \int_{O_t} \int_{O_w} \frac{e^{w(1-h)-ht}\, dw\, dt}{w^p t^{n-p+1}} = \frac{(-1)^{n-p}}{2\pi i} \int_{O_t} \frac{e^{-ht}(1-h)^{p-1}\, dt}{t^{n-p+1}(p-1)!} = \frac{h^{n-p}(1-h)^{p-1}}{(n-p)!\,(p-1)!}. \tag{4·7}$$

Now by definition the incomplete beta function

$$I_h(n-p+1,p) = n! \int_0^h t^{n-p}(1-t)^{p-1}\, dt / \{(n-p)!\,(p-1)!\},$$

therefore

$$H_h(n-p+1,p) = \frac{1}{n!} I_h(n-p+1,p) + \text{constant}. \tag{4·8}$$

To find the constant put $h = 1$ or 0. To find $H_0(n-p+1,p)$ the path of integration in the θ plane is split into two parts, one going (only) round $\theta = 0$ and the other round $\theta = w$. Integrating first with respect to θ, the first path gives $e^w w^{1-n}$ and the second $-e^w w^{1-n}$. Therefore $H_0(n-p+1,p) = 0$. Similarly, it can be verified that $H_1(n-p+1,p) = 1/n!$. Thus

$$H_h(n-p+1,p) = I_h(n-p+1,p)/n!. \tag{4·9}$$

Using (4·9) and (4·4) the first two terms of the expansion of the cumulative probability are found to be

$$P = I_h(n-p+1,p) + \frac{n(n-1)}{24M^2}\{(n+1)\,I_h(n-p+1,p) - (n-p+2)\,I_h(n-p-1,p)$$

$$- (p+1)\,I_h(n-p+1,p-2) + 2I_h(n-p,p-1)\}, \tag{4·10}$$

* This expansion for $\binom{N}{n}$ is the subject of a separate note (Wise, 1954).

in which form it is very easy to work out numerically on a machine, whenever we can choose values of h, n and p for which the incomplete beta functions have been tabulated (Pearson, 1934). The first term is the sum of the first p coefficients in the binomial series $\{h+(1-h)t\}^n$ of powers of t up to t^{p-1}. It looks as if the M^{-2} term diverges for larger n, and of course the first positive term and the two negative ones do so, but they tend to cancel one another. This is more obvious in the expansion for the inverse. However, another form of (4·10) is interesting. By working out the contour integrals in a different way, or by using recurrence relations connecting neighbouring values of the incomplete beta function,* we find that

$$P = I_h(n-p+1,p) + \frac{n!\,h^{n-p-1}(1-h)^{p-2}}{24M^2(n-p)!\,(p-1)!}\,\epsilon(h,n,p), \tag{4·11}$$

where $$\epsilon(h,n,p) = (n-p)(p-1)h(1-h)(2h-1) - (1-h^2)(1-h)\{(n-p+1)^2-1\} + (p^2-1)(2h^2-h^3). \tag{4·12}$$

Note that if p is replaced by $n-p+1$ and h by $1-h$, ϵ changes to $-\epsilon$, whilst its coefficient in (4·11) does not change. Further, $I_h(n-p+1,p)$ then becomes

$$I_{1-h}(p,n-p+1) \equiv 1 - I_h(n-p+1,p).$$

So P becomes $1-P$ exactly. But if h changes to $1-h$ and p to $n-p+1$, x becomes $1-x$, simply from the definition of h as in (3·8). This is easy to interpret, for if there are less than p black balls in the sample of n, with probability P, there are then at least $n-p+1$ white ones, and $1-P$ is the probability that there are less than $n-p+1$ white ones. This property of anti-symmetry is exactly satisfied by (4·11) and (4·12).

5. Expansion for the inverse

Given the probability P of finding less than p black balls in the sample of n, we wish to find the corresponding batch-proportion x. This occurs only in $h = \{N(1-x)-\frac{1}{2}n+\frac{1}{2}p\}/M$, where $M = N-\frac{1}{2}n+\frac{1}{2}$, and clearly the first approximation h_0 to h will be given by

$$I_{h_0}(n-p+1,p) = P. \tag{5·1}$$

It is clear too from (4·2) that we can write

$$h = h_0 + \delta/(24M^2) + O(M^{-4}). \tag{5·2}$$

Substituting (5·2) in (4·2), and neglecting the M^{-4} terms, which do not affect δ,

$$P = \left(\frac{1}{2\pi i}\right)^2\left\{1+\frac{(n-1)n(n+1)}{24M^2}\right\}\int_{O_\theta}\int_{O_w}\frac{e^{-h\theta+w}(w-\theta)^{p-n-1}}{\theta w^p} \times\left\{1-\frac{(n-p+1)(\theta-w)^2+pw^2+\theta^2}{24M^2}\right\}dw\,d\theta - \frac{\delta}{(2\pi i)^2\,24M^2}\int_{O_\theta}\int_{O_w}e^{-h_0\theta+w}(w-\theta)^{p-n-1}w^{-p}\,dw\,d\theta.$$

Hence

$$0 = \left(\frac{1}{2\pi i}\right)^2\int_{O_\theta}\int_{O_w}\frac{e^{-h_0\theta+w}}{\theta w^p(\theta-w)^{n-p+1}}[n^3-n-\{(n-p+1)(\theta-w)^2+pw^2+\theta^2\}-\theta\delta]\,dw\,d\theta. \tag{5·3}$$

* The coefficient of $n(n-1)/24M^2$ in (4·10) is expressed as the sum of three differences between incomplete beta functions, and these have coefficients $n-1$, 2 and $p+1$ respectively.

$|\theta| \gg |w|$ as before. Let us then consider the θ integration first. Its path goes round two poles, at $\theta = 0$ and $\theta = w$. So we replace this path by two paths, one round $\theta = 0$ only and the other round $\theta = w$ only. We call the two new integrals f_1 and f_2 respectively. For the first one we have $|\theta| < |w|$. Integrating round $\theta = 0$,

$$f_1 = \left(\frac{1}{2\pi i}\right)^2 \int_{\theta=0^+} \int_{O_w} \frac{e^w \{n^3 - n - w^2(n+1)\} d\theta\, dw}{w^p (-w)^{n-p+1} \theta} = \frac{1}{2\pi i} \int_{O_w} \frac{e^w}{w^{n+1}} \{(n+1)\, n(n-1) - (n+1) w^2\} dw. \tag{5·4}$$

Then expanding the exponent, the two terms from the residues at the origin are equal to $\pm(-1)^{n-p+1}(n+1)/(n-2)!$ respectively. Thus $f_1 = 0$ and we are left with the integral round $\theta = w$, and now we have $|\theta - w| < |w|$. Write equation (5·3) as

$$f_2 = 0 = \left(\frac{1}{2\pi i}\right)^2 \int_{C_\theta} \int_{O_w} \frac{e^{-h_0\theta + w}}{\theta w^p (\theta - w)^{n-p+1}} \Theta\, dw\, d\theta, \tag{5·5}$$

where $$\Theta = \{n^3 - n - w^2(n+1)\} + \{2w(n-p+1) - \delta\}\theta - \theta^2(n-p+2).$$

C_θ goes anti-clockwise round $\theta = w$. Write (5·5) as

$$0 = f_2 = f_3 + f_4 + f_5.$$

f_4 corresponds to the coefficient of θ in 'Θ' and is a multiple of the first derivative of an incomplete beta function with respect to h (see (4·6)) at $h = h_0$; f_5 is a multiple of the second derivative. To evaluate f_3 put $\theta = w(1+s)$. Then the s path of integration goes round the origin but not round $s = -1$. We get

$$f_3 = \left(\frac{1}{2\pi i}\right)^2 \int_{O_w} \int_{O_s} \frac{e^{w(1-h_0-h_0 s)}}{(1+s)\, w^{n+1} s^{n-p+1}} \{n^3 - n - w^2(n+1)\} ds\, dw. \tag{5·6}$$

Integrating first with respect to w we need only the residue at the origin. This gives

$$f_3 = \frac{(n+1)\, h_0}{(n-2)!\, 2\pi i} \int_{O_s} \frac{(1 - h_0 - h_0 s)^{n-2} (h_0 - 2 + h_0 s)\, ds}{s^{n-p+1}}.$$

Again from the residue at the origin, dropping the suffix 0 in h_0 from now on,

$$f_3 = \frac{(-1)^{n-p}(n+1)}{(n-p)!\,(p-1)!} (1-h)^{p-2} h^{n-p+1} \{(h-2)(p-1) - (1-h)(n-p)\}. \tag{5·7}$$

Now to evaluate f_4 and f_5 we use (4·6) and (4·7) and find that

$$f_4 = (-1)^{n-p} \left\{ \frac{2(n-p+1)\, h^{n-p} (1-h)^{p-2}}{(n-p)!\,(p-2)!} - \frac{h^{n-p}(1-h)^{p-1}}{(n-p)!\,(p-1)!} \delta \right\}, \tag{5·8}$$

$$f_5 = (-1)^{n-p} \frac{(n-p+2)\, h^{n-p-1} (1-h)^{p-2}}{(n-p)!\,(p-1)!} \{(n-p)(1-h) - (p-1)h\}. \tag{5·9}$$

The equation determining δ is $f_3 + f_4 + f_5 = 0$. After some simplification it appears that the best expression for δ is

$$\delta = \delta(h, n, p) = \left(\frac{1}{h} - h\right)(n-p+1)^2 + \left(1 - h - \frac{1}{1-h}\right) p^2 + (1-2h)\{(n-p)(p-1) - 1\} + \frac{1}{1-h} - \frac{1}{h}. \tag{5·10}$$

Recapitulating, when $h = \{N(1-x) - \frac{1}{2}n + \frac{1}{2}p\}/M$, $M = N - \frac{1}{2}n + \frac{1}{2}$, and x is such that

$$\sum_{r=0}^{p-1} P_{n,r} = P \text{ exactly},$$

$$h = h_0 + \delta(h_0, n, p)/(24M^2) + O(M^{-4}), \tag{5·11}$$

where $$I_{h_0}(n-p+1, p) = P.$$

Note that the anti-symmetry property is again exact, for

$$\delta(1-h, n, n-p+1) = -\delta(h, n, p). \tag{5·12}$$

In the next section we derive a simpler expression for δ.

6. A simple approximation to the second term in the expansion for the inverse function

The expression for δ can be surprisingly simplified for one of the most important practical applications. We have only to use an expansion for h_0, defined in (5·11) as an inverse incomplete beta function, in terms of percentage points of the χ^2 distribution (or of the Poisson distribution). We require the value of $y_0 = \frac{1}{2}\chi^2_{2p}(P)$, that is, half the value of χ^2 with $2p$ degrees of freedom and a probability P of being exceeded. The expansion is (Wise, 1950)

$$h_0 = \exp\frac{-y_0}{m}\left\{1 + \frac{(p-1)(p+1+y_0)}{24m^2} + O\left(\frac{p}{m}\right)^4\right\}, \tag{6·1}$$

where $m = n - \frac{1}{2}p + \frac{1}{2}$, or the mean of $n, n-1, \ldots, n-p+1$. Now in most practical sampling problems the batches have small proportions of defective units, and then, for all non-extreme probabilities P, y_0, which is of the same order of magnitude as p (see (6·4) below) is considerably smaller than n and m. So an expansion of δ in powers of m^{-1} should give a good approximation to δ. This was even better than expected, for it began with a term in m/y_0, and the next term was the coefficient of m^{-1}; there was no constant term. (I could not find any underlying reason why this is so.) As far as m^{-1} we find that

$$\delta(h_0, n, p) \doteqdot \frac{m}{y_0}(y_0 - p + 1)(2y_0 + p + 1) - 2m + \frac{y_0}{m}\delta_{-1}, \tag{6·2}$$

where $$\delta_{-1} = \frac{(y_0-p+1)(4y_0-7p+7)}{12} - 2 - \frac{(y_0-p-1)(2y_0+p+1)(p-1)^2}{24y_0^2}. \tag{6·3}$$

For many probabilities the y_0 values can be read directly from tables (Catherine Thompson 1941; Hald & Sinkbaek, 1950), so this formula is very easy to use. We can also see what happens to δ for large values of n, p, etc., although only very roughly. Campbell's (1923) expansion gives

$$y_0 = p + \xi_P\sqrt{p} + \tfrac{1}{3}(\xi_P^2 - 1) + O(p^{-\frac{1}{2}}), \quad \text{where} \quad \frac{1}{\sqrt{(2\pi)}}\int_{\xi_P}^{\infty} e^{-\frac{1}{2}t^2}\,dt = P, \tag{6·4}$$

and this gives

$$\delta \sim 3\frac{m}{y_0}\xi_P p^{\frac{1}{2}}, \quad \delta_{-1} \sim -\tfrac{3}{5}\xi_P p^{\frac{1}{2}}, \tag{6·5}$$

so that the δ_{-1} term is only of the order of $-y_0^2/(8m^2)$ times the leading term. More fully

$$\delta y_0/m = 3\xi_P p^{\frac{1}{2}} + 3\xi_P^2 p + \tfrac{1}{3}p^{\frac{1}{2}}(4\xi_P^3 - 3\xi_P) + \tfrac{1}{9}(2\xi_P^4 - \xi_P^2 + 8) \tag{6·5′}$$

So for a given n, δ increases with ξ_P and hence with $1-P$. In fact at the 50% point δ is very close to zero.

The correction term $\delta/(24M^2)$ is largest, for a given P, N and p, when $n = \frac{1}{2}N$—for $n > \frac{1}{2}N$ we can work with the unsampled units (see § 8(*b*)). When $n = \frac{1}{2}N$, $M \doteq \frac{3}{2}n$ and the term correcting h_0 is $\delta/(54n^2)$; asymptotically

$$\frac{\delta}{54n^2} \sim \frac{(n-\frac{1}{2}p+\frac{1}{2})\,\xi_P p^{\frac{1}{2}}}{18n^2}, \tag{6·6}$$

which tends to zero, although slowly, if p/n stays constant and both p and n tend to infinity. But the factor $\frac{1}{18}$ ensures that the correction term is always small.

7. Discussion

The accuracy of the sums of the first two terms of the expansions seems good enough for almost all practical purposes, as is shown by the numerical calculations in the next section; it is also sufficient for providing mathematical expressions in other problems where hypergeometric probabilities are needed. One term is enough when the sample size is rather smaller than 0·4 of the batch size (as in the example given here), or when both sample and batch are very large. The error in neglecting all other terms can then be estimated from (6·2) and (6·5). For extreme probabilities the inverse expansion is rather less accurate and for very extreme ones both expansions may break down. For we can write

$$1-h = \{Nx-\tfrac{1}{2}(p-1)\}/(N-\tfrac{1}{2}n+\tfrac{1}{2}). \tag{7·1}$$

Nx is the number of defectives in the batch, and if $Nx < p-1$ there can never be more than $p-1$ defectives observed, i.e. $P = 1$, whilst $1-h$ is not necessarily 1 (as it should be then) and can even be negative! Fortunately, these probabilities are seldom wanted and they often correspond to very few terms of the hypergeometric series and can be calculated or approximated to in another way.

We mentioned that in the formula for the direct probability, $1-h$ is in effect a proportion defective in a binomial sample of n, i.e. a sample from an infinite population. Let us write it as x_b. It is of interest that x_b can exactly equal the real proportion defective x, namely, if $x = (p-1)/(n-1)$ (when $p-1$ is nearly equal to the mean number of sampled defectives nx). This is perhaps clearer from the expansion for unknown x; we can rewrite (5·11) as

$$x = \left\{x_b - \frac{\delta(1-x_b, n, p)}{24(N-\frac{1}{2}n+\frac{1}{2})^2}\right\}\left(1-\frac{n-1}{2N}\right)+\frac{p-1}{2N}, \tag{7·2}$$

which gives a direct comparison between corresponding percentage points in binomial and hypergeometric distributions. The term containing δ is relatively small unless the sample size n is well over half the batch size N*.

It is surprising that such accurate but simple formulae should not have been found before; this may be because it is so often assumed that probabilities in a complex distribution function can only be simplified and calculated by fitting a simpler distribution function to it. Clearly this is not always true. A mathematical treatment with generating functions of two variables and double contour integrals seems novel in probability theory, but this type of expansion is well-known in statistical mechanics as a Borel or a Laplacian expansion.

* Hence we can nearly always use the simplest approximation to δ, neglecting δ_{-1} in (6·2). δ should then, strictly, be written $\delta(P, n, p)$.

8. Numerical examples

We shall give examples of the numerical calculation of the probability and of its inverse, and afterwards compare them with some exact values. In each case we shall calculate the whole of an 'operating characteristic'. This is defined as follows. Let a batch of N units be accepted if less than p defectives are observed in a sample of n from the batch. Let Nx be the number of defectives in a batch; then the probability P of accepting the batch decreases as x increases and the P, x curve is the operating characteristic; the P values are sums of the first p terms of a hypergeometric series (as in (1·6)). Obviously P can be calculated for given values of x or x as a function of P; in both cases it is convenient to suppose that Nx can take all values between 0 and N.

(a) *Example with unknown probabilities*

We shall calculate the probability of accepting a batch of $N = 200$, sample size $n = 80$, with $p = 33$.* This (see (4·10)) is obtained from

$$P = \sum_{r=0}^{p-1} P_{n,r} = I_h(n-p+1, p) + \frac{(n-1)n}{24M^2} I_2, \tag{8·1}$$

where $M = N - \frac{1}{2}n + \frac{1}{2}$, $h = \{N(1-x) - \frac{1}{2}n + \frac{1}{2}p\}/M$,

$$I_2 = (n+1) I_h(n-p+1, p) - (n-p+2) I_h(n-p-1, p) - (p+1) I_h(n-p+1, p-2) + 2I_h(n-p, p-1),$$

and $I_h(n-p+1, p)$ is the incomplete beta function, which here gives the binomial probability of observing less than p defectives in a sample of n from a population with proportion-defective equal to $1-h$.

The obvious thing to do is to calculate P for a whole series of values of $1-h$, and I have done this for the sixteen equally spaced values $1-h = 0{\cdot}46\,(0{\cdot}02)\,0{\cdot}76$. All the incomplete beta functions required can be read directly from Pearson's (1934) tables; further, the correction term I_2 is so quickly obtained using a calculating machine that only the first incomplete beta function needs to be written down. For the example chosen with

$$N = 200, \quad n = 80, \quad p = 33, \quad M = 160{\cdot}5,$$

we have

$$\sum_{0}^{p-1} P_{n,r} = I_h(48, 33) + \frac{79 \times 80}{24(160{\cdot}5)^2}\{81 I_h(48, 33) - 49 I_h(46, 33) - 34 I_h(48, 31) + 2 I_h(47, 32)\}. \tag{8·2}$$

The effective numbers of defectives are given by

$$200x = 176{\cdot}5 - 160{\cdot}5h. \tag{8·3}$$

Corresponding to these, the probabilities of acceptance are set out in Table 1.

* This is by no means unrealistic, although 32 defectives out of 80 would be a very high proportion. But it is sometimes advantageous to test units by a gauge with closer limits than the tolerance limits prescribed for the units themselves; for example, v.d. Weiden (1953) has convincingly demonstrated how efficiency is thereby improved in many attribute sampling systems for inspection and quality control of batches.

(b) Numerical accuracy of example under (a)

By interpolating in Table 1 as accurately as possible (to do this the probabilities were first transformed to probits), I obtained the probabilities P_1 given in Table 2; under these are set out the corresponding exact values of P (v. Heezwijk, v.d. Linden & v. Strik, 1947).

Table 1. *Numerical values of probabilities calculated from one and from two terms of* (8·2), *as a continuous function of the proportion defective* x ($N = 200$, $n = 80$, $p = 33$)

$200x$	54·52	57·73	60·94	64·15	67·36	70·57	73·78	76·99
P (1st term only)	0·99951	0·99789	0·99264	0·97886	0·94879	0·89343	0·80634	0·68808
P (1st two terms)	·99975	·99863	·99449	·98254	·95471	·90109	·81410	·69365
$200x$ (cont.)	80·20	83·41	86·62	89·83	93·04	96·25	99·46	102·67
P (1st term only)	0·54837	0·40392	0·27264	0·16745	0·09300	0·04646	0·02075	0·00824
P (1st two terms)	·54989	·40095	·26636	·15992	·08613	·04133	·01752	·00651

Table 2. *Comparison of probabilities* P_1 *obtained by interpolation in Table* 1 *(first two terms of expansion) with exact values*

$200x$	100	90	80	70	60
P_1	0·01677	0·15518	0·55922	0·91289	0·99623
P (exact)	·01504	·15494	·55929	·91308	·99614

(c) Calculating the probabilities when the sample size is more than half the batch size

If more than half the units in the batch are sampled it is more accurate (although less convenient) to calculate the corresponding accumulated probabilities for the unsampled units. If the sample number is now n' ($> \frac{1}{2}N$) and P' is the probability that $p'-1$ defectives or less are found when there are Nx' defectives in the batch, elementary reasoning like that at the end of § 4 shows that formula (8·1) is still applicable, but with

$$n = N-n', \quad p = N(1-x)-n'+p', \quad x = 1-x', \quad P = P'.$$

Hence $M = \frac{1}{2}(N+n'+1)$, $\quad n-p+1 = Nx'-p'$, $\quad h = (p'+Nx')/(N+n'+1)$.

Obviously this result breaks down if $Nx' < p'$ or $p < 0$, but then the corresponding probabilities are respectively 1 and 0.

(d) Calculation when the incomplete beta values are not tabulated

Outside the range of Pearson's tables calculating the probabilities is less simple and the inverse function is generally easier to find. If we do need the direct probabilities, it is best to use formulae like the following (Wise, 1950):

$$I_h(n-p+1,p) = e^{-y}\left\{1+y+\frac{y^2}{2!}+\ldots+\frac{y^{p-1}}{(p-1)!}\right\}+\frac{e^{-y}(p^3-p)}{24m^2}\left\{\frac{y^p}{p!}+\frac{y^{p+1}}{(p+1)!}\right\}+\ldots,$$

where $m = n - \frac{1}{2}p + \frac{1}{2}$, $y = -m \log_e h$. The first term is a cumulative sum of Poisson probabilities and the second term the difference between two such sums; both can be obtained (usually with at most one-dimensional interpolation) from Molina's (1945) tables or those of Pearson & Hartley (1954, Table 7).

(e) *Calculation of the inverse function*

We take the same batch size N and sample size n, but p equal to 7 instead of 33, and seek the proportion-defective x such that the probability of observing less than p defectives in the sample of n is P. In general terms, that is, irrespective of whether Nx is a whole number or not, we ask what is the value of x such that the corresponding sum of the first p hypergeometric probabilities has a ratio P to the sum of all of them ($n+1$ of them)? Obviously the first approximation $x = x_0$ is given by

$$I_{h_0}(n-p+1, p) = P, \tag{8·4}$$

where

$$Nx = (N - \tfrac{1}{2}n + \tfrac{1}{2})(1-h) - \tfrac{1}{2}(p-1), \tag{8·5}$$

and $x = x_0$ when $h = h_0$ (see also (7·2)). The second approximation, allowing for the M^{-2} term, where again $M = N - \frac{1}{2}n + \frac{1}{2}$, is

$$h = h_0 + \delta/(24M^2). \tag{8·6}$$

Table 3. *Values of equivalent numbers of defectives for given probabilities*; $N = 200$, $n = 80$, $p = 7$. $200x_0$ *calculated from first term of* (8·6), $200x_1$ *from first two terms*, $200x$ *by interpolation from exact probabilities*

P ...	0·995	0·990	0·975	0·950	0·900	0·750	0·500
$200x_0$	7·196	7·791	8·768	9·712	10·925	13·264	16·334
$200x_1$	7·544	8·222	9·117	9·990	11·134	13·367	16·330
$200x$	—	8·24	9·145	9·993	11·138	13·366	16·318

P ...	0·250	0·100	0·050	0·025	0·010	0·005
$200x_0$	19·904	23·542	25·907	28·068	30·703	32·571
$200x_1$	19·796	23·332	25·634	27·733	30·306	32·097
$200x$	19·784	23·323	25·622	—	—	—

For δ, see equation (5·10); however, in this particular case δ can be simplified (see § 6 and, in particular, (6·2) for full details) to

$$\delta = m\left\{\frac{(y_0 - p + 1)(2y_0 + p + 1)}{y_0} - 2\right\}, \tag{8·7}$$

where $m = n - \frac{1}{2}p + \frac{1}{2}$, $y_0 = \frac{1}{2}\chi^2_{2p}(P)$.

Again x will be calculated for a whole range of probabilities; we therefore require values of h_0 such that

$$I_{h_0}(74, 7) = P.$$

These were obtained (see equation (6·1)) from the close approximation

$$h_0 = \exp\left[\frac{-y_0}{77}\left\{1+\frac{6(8+y_0)}{24\times 77^2}\right\}\right]. \qquad (8\cdot 8)$$

Here the values of $2y_0$ are χ^2 values with $2p = 14$ degrees of freedom and a probability P of being exceeded. These were read off from Catherine Thompson's tables (1941) for all the probabilities given there. The resulting first and second approximations are x_0 and x_1 and the corresponding effective numbers of defectives, i.e. 200 times these, are given in Table 3. The last row of the table shows, as $200x$, some exact values obtained by inverse interpolation from the exact probabilities calculated by v. Heezwijk *et al.* (1947); these again were first transformed to probits and differences up to the fourth allowed for.

We see that the first approximation is appreciably different from the second one except near $P = \frac{1}{2}$; of course this difference decreases rapidly with the ratio (n/N) of sample size to batch size.

REFERENCES

CAMPBELL, G. A. (1923). Probability curves showing Poisson's exponential summation. *Bell. Syst. Tech. J.* **2**, 97.

DAVIES, O. L. (1933). On asymptotic formulae for the hypergometric series. I. Hypergeometric series in which the fourth term is unity. *Biometrika*, **25**, 295.

FINNEY, D. J. (1948). The Fisher-Yates test of significance in 2×2 contingency tables. *Biometrika*, **35**, 145.

HALD, A. & SINKBAEK, S. A. (1950). A table of percentage points of the χ^2 distribution. *Skand. ActuarTidskr.* **33**, 168.

v. HEEZWIJK, A., v.D. LINDEN, H. & v. STRIK, R. (1947). Unpublished calculations in Philips Research Laboratories, Eindhoven.

KATZ, L. (1953). Confidence intervals for the number showing a certain characteristic in a population when sampling is without replacement. *J. Amer. Statist. Ass.* **48**, 256.

MOLINA, E. C. (1945). *Tables of the Poisson exponential.* D. van Nostrand and Co.

PEARSON, KARL (1934). *Tables of the Incomplete Beta Function.* Cambridge: the University Press for the Biometrika Trust.

PEARSON, E. S. & HARTLEY, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge: the University Press for the Biometrika Trust.

THOMPSON, CATHERINE (1941). Tables of percentage points of the χ^2 distribution. *Biometrika*, **32**, 187.

v.D. WEIDEN, H. (1953). Cost factors and the choice of a statistical inspection scheme (in Dutch). Thesis, Delft, and Memo. from the Technical Efficiency and Organization Department, Philips, Eindhoven.

WISE, M. E. (1950). The incomplete beta function as a contour integral and a quickly converging series for its inverse. *Biometrika*, **37**, 208.

WISE, M. E. (1954). The ratio of two factorials and some fundamental probabilities (to be published).

YATES, F. (1934). Contingency tables involving small numbers and the χ^2 test. *Suppl. J.R. Statist. Soc.* **1**, 217.

THE QUOTIENT OF A RECTANGULAR OR TRIANGULAR AND A GENERAL VARIATE

By S. R. BROADBENT

British Coal Utilisation Research Association

1. Introduction

It is common to take measurements of two quantities in order to determine their ratio. For instance, the input and output of work determine the *efficiency* of an engine; the time taken to traverse a distance measures *velocity*; and the residue of a weight of reactant in a chemical process gives the *proportion* which has reacted. Efficiency, velocity, proportion and indices in biometry are examples of quantities which are necessarily ratios. When the two measurements made are subject to error, their quotient is distributed about its expected value.

There are cases in which experience, *a priori* considerations, or repeated trials give the distributions of error of the numerator and denominator separately. It may, however, be impossible or undesirable for reasons of economy to repeat the experiment in which the two quantities are measured simultaneously. The precision of a single trial must be found by theoretical arguments. Two examples will demonstrate this. The acceptance trial of an industrial boiler is a lengthy and expensive process which results in a figure for efficiency. It is necessary, from this one experiment, to state confidence limits to the true value of the efficiency. Secondly, the reading of a flow-meter in a pipeline may depend on the ratio of two quantities, whose dimensions have been manufactured within certain tolerances. It is required to state, without the great number of experiments needed to calibrate a sample of these meters, the tolerance of the reading of a typical meter.

It may be the fiducial distribution of the quotient that is required, or the frequency distribution; for the types discussed the two are formally identical. Those limits which are obtained by taking the extreme possible values of numerator and denominator are unnecessarily wide. Approximate methods, such as the assumption of the normality of the quotient, may suffice for some purposes, but these methods should when possible be compared with the exact results, which it is clearly desirable to have.

It often happens that errors of measurements are distributed in one of two ways: in rectangular distributions or in normal (Gaussian) distributions. Instrumental error is often rectangular; length, time and weight are usually recorded 'plus or minus X', it being understood that the true value is equally likely to lie anywhere between these limits. It may be noted that replication is often pointless in these cases; the instrument is incapable of recording with greater precision and repeated readings will give identical results. A measurement which has been made precisely and is then rounded off is also subject to rectangular error. Sampling error, and the sum or mean of a number of errors, may often be taken as normal.

A measurement is sometimes the difference of two readings, each of which is liable to independent errors from the same distribution. The increase in weight during a certain process, and the measurement of a length (or other quantity) by recording the distances of

its ends from an arbitrary zero, are examples of this type of measurement. The difference of two rectangular variates with the same range is a triangular variate; the difference of two normal variates is itself normally distributed.

There are therefore three common distributions of error of measurement: *rectangular*, *triangular* and *normal*. These three types of error define six distinct distributions of quotient, the order in which a pair of types is chosen being immaterial, since the distribution of the reciprocal of a quotient is determined by that of the quotient. The six distributions may be represented, in an obvious notation, by R/R, R/T, T/T, R/N, T/N and N/N. The first three distributions are not difficult to obtain; the distribution N/N has been discussed by Geary (1930), Fieller (1932), Nicholson (1941) and Creasy (1954). The remaining distributions, R/N and T/N, form the subject of the present paper.

These distributions may also be used for testing two variates, rectangular, triangular, or normal, said to have the same mean or to have means in a given ratio.

The quotients R/N and T/N have frequency and distribution functions which can be got by straightforward integration. The expressions obtained are difficult to manipulate and awkward for numerical work. The method described below obtains the distributions in these and more general cases in a form which makes numerical work possible. In particular, existing tables assist the calculation of percentage points of the distribution.

2. THE QUOTIENT OF A RECTANGULAR AND A GENERAL VARIATE

We require the distribution function of $q' = (a+b\eta)/(\mu+\sigma\xi)$, where η is rectangularly distributed in $(-1, 1)$ and ξ is independent of η with mean zero and variance 1. We suppose $a>0$, $b/a = \alpha$, $0<\alpha<1$, $\mu>0$, $\sigma/\mu = \beta$, $0<\beta<\infty$. The distribution of ξ is also subject to the restriction given below. Since $q' = (a/\mu)(1+\alpha\eta)/(1+\beta\xi)$, we consider the standardized quotient $q = \mu q'/a = y/x$, where y is rectangularly distributed in $(1-\alpha, 1+\alpha)$ and x has mean 1 and variance β^2. If x has the distribution function $F(x)$, we suppose the frequency function $(d/dx)F(x) = f(x)$ exists almost everywhere, and that $F(x) = z$ has inverse $x = G(z)$ defined, non-zero, and finite for almost all z. This condition precludes mass points at $x = 0$ and $x = \infty$; it is satisfied by all the usual distribution functions. Let $F(0) = \epsilon$; we do not at present suppose ϵ small.

Now z is independent of y; z is distributed rectangularly in $(0, 1)$. The joint distribution of z and y is $dz\,dy/(2\alpha)$. For each z, $dy = |G(z)|\,dq$, and q has the range

$$((1-\alpha)/G(z),\ (1+\alpha)/G(z)) \quad \text{if} \quad z>\epsilon,$$
$$((1+\alpha)/G(z),\ (1-\alpha)/G(z)) \quad \text{if} \quad z<\epsilon.$$

Therefore z and q have the joint distribution $|G(z)|\,dz\,dq/(2\alpha)$ over the regions shown in Fig. 1.

$\Pr(0<q\leqslant Q_1)$ $(Q_1>0)$ is given by the integral of this element over the region bounded by $q = Q_1$, $z = 1$ and $q = (1-\alpha)/G(z)$ *minus* the integral over the region bounded by $q = Q_1$, $z = 1$ and $q = (1+\alpha)/G(z)$. When we have integrated the repeated integral with respect to q the first integral is

$$\frac{1}{2\alpha}\int_{z=F[(1-\alpha)/Q_1]}^{1} \{Q_1 G(z) - (1-\alpha)\}\,dz.$$

Changing the variable to ξ, and writing

$$g(\xi) = \beta f(1+\beta\xi),$$

the frequency function of the standardized form of x, and

$$X = (1-\alpha-Q_1)/(\beta Q_1),$$

we write this integral

$$\frac{\beta Q_1}{2\alpha}\int_{\xi=X}^{\infty}(\xi-X)\,g(\xi)\,d\xi.$$

Writing

$$J_1 g(X) = \int_{\xi=X}^{\infty}(\xi-X)\,g(\xi)\,d\xi,$$

and

$$Y = (1+\alpha-Q_1)/(\beta Q_1),$$

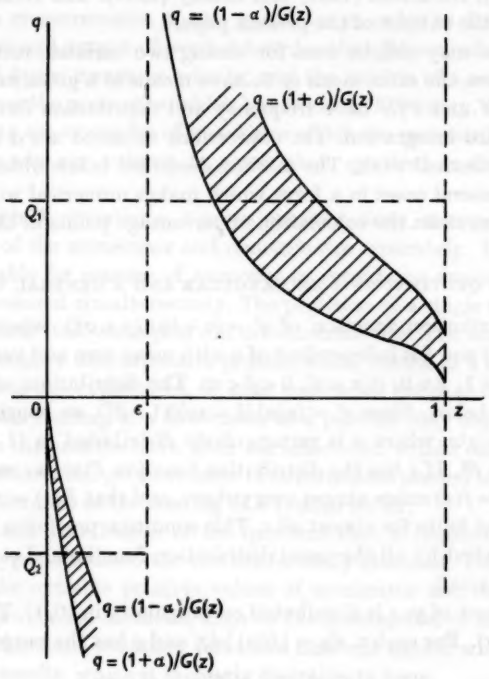


Fig. 1

and remembering that $\Pr(q \leqslant 0) = \Pr(x < 0) + \Pr(x = \infty) = \epsilon$, we see that the cumulative function of q is, for $Q_1 > 0$,

$$\Pr(q \leqslant Q_1) = \epsilon + \frac{\beta Q_1}{2\alpha}\{J_1 g(X) - J_1 g(Y)\}. \tag{1}$$

It follows from (1) that

$$\Pr(q > Q_1) = \frac{\beta Q_1}{2\alpha}\{[Y + J_1 g(Y)] - [X + J_1 g(X)]\} - \epsilon. \tag{2}$$

These forms are particularly useful when ϵ is small or zero for the calculation of percentage points at the tails of the distribution of q. For as $Q_1 \to 0$ and $X, Y \to \infty$, both $J_1 g(X)$ and $J_1 g(Y) \to 0$, the former more slowly than the latter. For small Q_1, i.e. at lower percentage points, $J_1 g(Y)$ is small in comparison with $J_1 g(X)$ for many distributions (in par-

ticular when x is normal). In these cases, by successive approximation in X, lower percentage points may be calculated from (1) when a table of $J_1 g(X)$ exists. Similarly, for large Q_1, i.e. at upper percentage points, $X + J_1 g(X)$ is small in comparison with $Y + J_1 g(Y)$ for many distributions. From (2) upper percentage points of q may therefore be calculated. The percentage points of q' are (a/μ) times those of q.

When ϵ is not small, we may consider $\Pr(q \leqslant Q_2)$ $(Q_2 < 0)$. The integrals over the regions bounded by $q = Q_2$, $z = \epsilon$, $q = (1+\alpha)/G(z)$ and $q = Q_2$, $z = \epsilon$, $q = (1-\alpha)/G(z)$ may be expressed in the same way as

$$\Pr(q \leqslant Q_2) = \frac{\beta Q_2}{2\alpha}\left\{\int_{\xi=Y}^{-1/\beta}(\xi - Y)\, g(\xi)\, d\xi - \int_{\xi=X}^{-1/\beta}(\xi - X)\, g(\xi)\, d\xi\right\},$$

where
$$Y = (1+\alpha-Q_2)/(\beta Q_2), \quad X = (1-\alpha-Q_2)/(\beta Q_2).$$

3. The quotient of a triangular and a general variate

The sum (or difference) of two independent rectangular distributions of the same half-range b is triangularly distributed. If the sum (or difference) of their means is a, we consider the distribution of $a + b\eta$, where η is distributed triangularly in $(-2, 2)$. We require the distribution of $q' = (a+b\eta)/(\mu+\sigma\xi)$. We suppose $a > 0$, $b/a = \alpha$, $0 < 2\alpha < 1$, $\mu > 0$, $\sigma/\mu = \beta$, $0 < \beta < \infty$. We suppose also that ξ is independent of η and satisfies the conditions of § 2. We consider the standardized quotient $q = \mu q'/a = y/x$. Here y has the frequency element

$$(y - 1 + 2\alpha)/(4\alpha^2), \quad (1 - 2\alpha \leqslant y \leqslant 1),$$
$$(1 + 2\alpha - y)/(4\alpha^2), \quad (1 \leqslant y \leqslant 1 + 2\alpha).$$

The joint distribution of $z = F(x)$ and q is

$$G(z)\{qG(z) - 1 + 2\alpha\}\, dz\, dq/(4\alpha^2), \quad ((1-2\alpha)/G(z) \leqslant q \leqslant 1/G(z)),$$
$$G(z)\{1 + 2\alpha - qG(z)\}\, dz\, dq/(4\alpha^2), \quad (1/G(z) \leqslant q \leqslant (1+2\alpha)/G(z)),$$

when $z > \epsilon$, and corresponding expressions when $z < \epsilon$, over the regions shown in Fig. 2.

$\Pr(0 < q \leqslant Q_1)$ $(Q_1 > 0)$ is the integral of the first element over the region bounded by $q = Q_1$, $z = 1$ and $q = (1-2\alpha)/G(z)$, *minus* the integral of the difference of the two elements over the region bounded by $q = Q_1$, $z = 1$ and $q = 1/G(z)$, *minus* the integral of the second element over the region bounded by $q = Q_1$, $z = 1$ and $q = (1+2\alpha)/G(z)$.

The first of these integrals is

$$\frac{1}{4\alpha^2}\int_{z=F[(1-2\alpha)/Q_1]}^{1} \tfrac{1}{2}\{Q_1^2[G(z)]^2 - 2Q_1 G(z)(1-2\alpha) + (1-2\alpha)^2\}\, dz.$$

When we change the variable to ξ and write

$$U = (1 - 2\alpha - Q_1)/(\beta Q_1),$$

the integral becomes

$$\frac{\beta^2 Q_1^2}{4\alpha^2}\frac{1}{2}\int_{\xi=U}^{\infty}(\xi - U)^2 g(\xi)\, d\xi.$$

Writing
$$J_2 g(U) = \frac{1}{2!}\int_{\xi=U}^{\infty}(\xi - U)^2 g(\xi)\, d\xi,$$
$$V = (1 - Q_1)/(\beta Q_1) \quad \text{and} \quad W = (1 + 2\alpha - Q_1)/(\beta Q_1),$$

we obtain the full expression

$$\Pr(q \leqslant Q) = \epsilon + \frac{\beta^2 Q^2}{(2\alpha)^2}\{J_2 g(U) - 2J_2 g(V) + J_2 g(W)\}. \tag{3}$$

It follows from (3) that

$$\Pr(q>Q)=\frac{\beta^2Q^2}{(2\alpha)^2}\left\{\left[\frac{U^2+1}{2}-J_2g(U)\right]-2\left[\frac{V^2+1}{2}-J_2g(V)\right]+\left[\frac{W^2+1}{2}-J_2g(W)\right]\right\}-\epsilon. \tag{4}$$

The forms (3) and (4) have the advantages in calculation noted for (1) and (2) respectively. When ϵ is not small $\Pr(q\leqslant Q_2)$ $(Q_2<0)$ may be obtained as before.

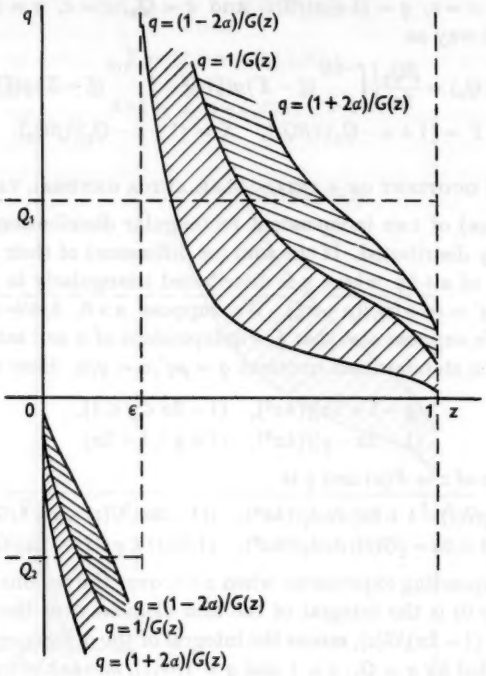


Fig. 2

4. The general theorem

The results given for one rectangular variate and for the sum of two independent rectangular variates of the same range can be generalized for the sum of n independent rectangular variates of the same range, when the resultant distribution is that given by Irwin (1927) and Hall (1927).

Let y have the Irwin-Hall distribution, i.e.

$$y=y_1+\ldots+y_n \quad (n=1,2,\ldots),$$

where each y_i $(i=1,\ldots,n)$ is a rectangular variate independent of $y_1,\ldots,y_{i-1}$ and with range (a_i-b,a_i+b). Let $\sum_{i=1}^{n}a_i=a>0$, $b/a=\alpha$, $0<n\alpha<1$. Let x be independent of y and have the distribution function $F(x)$ such that (i) x has mean $\mu>0$ and variance σ^2, $\sigma/\mu=\beta$, $0<\beta<\infty$, (ii) the frequency function $(d/dx)F(x)=f(x)$ exists almost everywhere, and (iii) $F(x)=z$ has inverse $x=G(z)$ defined, non-zero, and finite for almost all z. Let $F(0)=\epsilon$ and $Q>0$.

Then if $q = y/x$, $\Pr(q \leqslant aQ/\mu)$ is

$$\epsilon + \left(\frac{\beta Q}{2\alpha}\right)^n \left\{ J_n g(X_0) - \binom{n}{1} J_n g(X_1) + \binom{n}{2} J_n g(X_2) - \ldots \right\},$$

the series terminating at the $(n+1)$th term, where

$$J_n g(X) = \frac{1}{n!} \int_{\xi=X}^{\infty} (\xi - X)^n g(\xi)\, d\xi,$$

$$g(\xi) = \beta f(1 + \beta\xi),$$

and

$$X_r = (1 - (n-2r)\alpha - Q)/(\beta Q) \quad (r = 0, 1, \ldots, n).$$

5. THE FUNCTIONS $J_n g(X)$

$J_n g(X)$ is the Riemann-Liouville integral of the nth order; it may also be written

$$\int_{\xi_n=X}^{\infty} \int_{\xi_{n-1}=\xi_n}^{\infty} \ldots \int_{\xi=\xi_1}^{\infty} g(\xi)\, d\xi\, d\xi_1 \ldots d\xi_{n-1} d\xi_n.$$

From this form it is clear that $\lim_{X\to\infty} J_n g(X) = 0$, and that $(d/dX) J_n g(X) = -J_{n-1} g(X)$. This relation between the derivate and the integral of lower order may be used for interpolation in tables of $J_n g(X)$.

It is easy to show that $X + J_1 g(X) = \int_{\xi=-X}^{\infty} (\xi + X)\, g(-\xi)\, d\xi$. It follows that

(i) $\lim_{X\to-\infty} [X + J_1 g(X)] = 0$, and

(ii) if $g(\xi)$ is symmetrical about $\xi = 0$, i.e. if $f(x)$ is symmetrical about $x = 1$, then $X + J_1 g(X) = J_1 g(-X)$. Similarly, it may be shown that

(i) $\lim_{X\to-\infty} \left[\frac{X^2+1}{2} - J_2 g(X)\right] = 0$,

(ii) if $f(x)$ is symmetrical about $x = 1$, then $\frac{X^2+1}{2} - J_2 g(X) = J_2 g(-X)$, and

(iii) $\left(\frac{d}{dX}\right)\left[\frac{X^2+1}{2} - J_2 g(X)\right] = X + J_2 g(X)$.

The pairs of forms, (1) and (2), (3) and (4), therefore have symmetrical properties.

When x is distributed normally, $J_n g(X) = H h_n(X)/\sqrt{(2\pi)} = I_n(x)$, the Hermitian probability integral of the nth order, tabulated in the *British Association Tables* (1931, 1946). An account of these functions has been given by Fisher (*B.A. Tables*, 1931). Since the normal distribution is symmetrical about its mean, these tables may be used for $X + I_1(X)$ and $\frac{1}{2}(X^2+1) - I_2(X)$ by changing the sign of the argument.

6. TABULATION OF PERCENTAGE POINTS

When x is distributed normally and β is small (less than 0·25) ϵ is negligible (less than $1/10^4$). In the tables below (1) has been used to calculate the 1 and 5 % points for the distribution of the standardized quotient of a rectangular variate with range $(1-\alpha, 1+\alpha)$ and a normal variate $(1, \beta^2)$ for $\alpha = 0(0{\cdot}02)0{\cdot}10$ and $\beta = 0(0{\cdot}01)0{\cdot}05$. Similarly (2) has been used for the 95 and 99 % points, and (3) and (4) have been used for the quotient of a triangular and a normal variate, $\alpha = 0(0{\cdot}01)0{\cdot}05$ and $\beta = 0(0{\cdot}01)0{\cdot}05$. The percentage points of the quotients before standardization are given by multiplication by (a/μ). The percentage

Table 1. *Percentage points of the quotient of a rectangular and an independent normal variate. If y is rectangular with range $(1-\alpha, 1+\alpha)$, and x is normal with mean 1 and variance β^2, the table gives percentage points of (y/x)*

1 % Points

100β \ 100α	0	2	4	6	8	10
0	1·000	0·980	0·961	0·941	0·922	0·902
1	·977	·967	·951	·933	·915	·896
2	·956	·949	·935	·919	·903	·885
3	·935	·930	·919	·904	·889	·873
4	·915	·911	·902	·889	·875	·859
5	·896	·893	·885	·873	·860	·846

5 % Points

100β \ 100α	0	2	4	6	8	10
0	1·000	0·982	0·964	0·946	0·928	0·910
1	·984	·975	·960	·944	·927	·909
2	·968	·963	·951	·937	·921	·905
3	·953	·950	·940	·928	·914	·898
4	·938	·936	·928	·917	·905	·891
5	·924	·922	·916	·906	·895	·882

95 % Points

100β \ 100α	0	2	4	6	8	10
0	1·000	1·018	1·036	1·054	1·072	1·090
1	1·017	1·025	1·040	1·057	1·074	1·091
2	1·034	1·039	1·051	1·066	1·081	1·098
3	1·052	1·055	1·065	1·078	1·092	1·107
4	1·070	1·073	1·080	1·091	1·105	1·119
5	1·090	1·092	1·098	1·108	1·119	1·133

99 % Points

100β \ 100α	0	2	4	6	8	10
0	1·000	1·020	1·039	1·059	1·078	1·098
1	1·024	1·034	1·051	1·069	1·087	1·106
2	1·049	1·055	1·070	1·086	1·104	1·122
3	1·075	1·080	1·092	1·107	1·123	1·140
4	1·103	1·106	1·116	1·130	1·145	1·162
5	1·132	1·135	1·143	1·155	1·169	1·185

Table 2. *Percentage points of the quotient of a triangular and an independent normal variate. If y is triangular with range $(1-2\alpha, 1+2\alpha)$, and x is normal with mean 1 and variance β^2, the table gives percentage points of (y/x)*

1 % Points

100β \ 100α	0	1	2	3	4	5
0	1·000	0·983	0·966	0·948	0·931	0·914
1	·977	·971	·958	·943	·927	·911
2	·956	·952	·943	·931	·917	·902
3	·935	·932	·925	·916	·904	·891
4	·915	·913	·908	·899	·889	·878
5	·896	·894	·890	·883	·874	·864

5 % Points

100β \ 100α	0	1	2	3	4	5
0	1·000	0·986	0·973	0·959	0·945	0·932
1	·984	·979	·969	·956	·943	·930
2	·968	·965	·959	·949	·938	·926
3	·953	·951	·946	·939	·929	·919
4	·938	·937	·933	·927	·919	·910
5	·924	·922	·919	·915	·908	·900

95 % Points

100β \ 100α	0	1	2	3	4	5
0	1·000	1·014	1·027	1·041	1·055	1·068
1	1·017	1·021	1·032	1·044	1·057	1·070
2	1·034	1·037	1·043	1·053	1·064	1·076
3	1·052	1·054	1·059	1·066	1·076	1·086
4	1·070	1·072	1·075	1·082	1·090	1·099
5	1·090	1·091	1·094	1·099	1·106	1·114

99 % Points

100β \ 100α	0	1	2	3	4	5
0	1·000	1·017	1·034	1·052	1·069	1·086
1	1·024	1·030	1·043	1·058	1·074	1·090
2	1·049	1·052	1·062	1·074	1·088	1·103
3	1·075	1·078	1·085	1·094	1·106	1·120
4	1·103	1·105	1·110	1·118	1·129	1·140
5	1·132	1·134	1·138	1·145	1·154	1·164

points of $1/q$ are naturally the reciprocals of complementary percentage points of q. When $\alpha = 0$ we have the reciprocal of a normal variate, and when $\beta = 0$ we have a rectangular or triangular variate. The third decimal place in the tables should never be in error by more than one.

7. EXAMPLE OF THE USE OF THE DISTRIBUTION

The efficiency of a steam boiler may be estimated by burning a measured weight of coal of known calorific value, and dividing the heat obtained from the boiler by the heat supplied. The formula used for percentage efficiency is $E = kW/C$, where k is a constant in a particular test, W is the weight of water evaporated in the test, and C the weight of coal burned in the same period. The water supplied to the boiler during the test is measured precisely, but the water level in the boiler at the beginning and end of the test must be estimated from the readings of a water gauge with rectangular error. Suppose that k in a particular short test is 11·7, W is estimated as 3750 lb. with triangular error of range $\pm$ 300 lb. (the difference of two rectangular errors of range $\pm$ 150 lb.), and C is estimated as 600 lb. with a normal error of S.D. 12 lb., E is therefore calculated as 73·125 %. We are required to set confidence limits to this efficiency.

Here $\alpha = 0{\cdot}04$ and $\beta = 0{\cdot}02$; these parameters indicate the relative precision with which C and W are measured. The 1, 5, 95 and 99 percentage points of the distribution of the standardized quotient are

$$0{\cdot}917, \quad 0{\cdot}938, \quad 1{\cdot}064, \quad 1{\cdot}088.$$

These are to be multiplied by (ka/μ), i.e. the corresponding confidence limits to E are

$$67{\cdot}1, \quad 68{\cdot}6, \quad 77{\cdot}8, \quad 79{\cdot}6\,\%.$$

A 90 % confidence interval for E is therefore

$$68{\cdot}6 - 77{\cdot}8\,\%.$$

The author wishes to thank Mr R. L. Brown for suggesting the problem, Prof. G. A. Barnard for his assistance in the preparation of this paper, and the British Coal Utilisation Research Association for permission to publish it.

REFERENCES

British Association Mathematical Tables, **1** (1931), 1st ed. Cambridge University Press.
British Association Mathematical Tables, **1** (1946), 2nd ed. Cambridge University Press.
CREASY, M. A. (1954). *J.R. Statist. Soc.* B (in the Press).
FIELLER, E. C. (1932). *Biometrika*, **24**, 428.
GEARY, R. C. (1930). *J.R. Statist. Soc.* A, **93**, 442.
HALL, P. (1927). *Biometrika*, **19**, 240.
IRWIN, J. O. (1927). *Biometrika*, **19**, 225.
NICHOLSON, C. (1941). *Biometrika*, **32**, 16.

AN ESTIMATION PROBLEM IN QUANTITATIVE ASSAY

BY J. AITCHISON AND J. A. C. BROWN

Department of Applied Economics, University of Cambridge

1. THE PROBLEM

The purpose of this note is to show how a simple but common type of heteroscedasticity may be treated explicitly in quantitative assay.

Finney (1947) considered the problem of quantitative response to stimuli in biological assay and presented an estimation procedure based on the method of maximum likelihood. If u measures the quantitative response to a stimulus of concentration x, measured on a suitable scale, then the model used by Finney is essentially

$$u = HP(\alpha+\beta x)+\epsilon, \tag{1}$$

where

$$P(\alpha+\beta x) = \int_{-\infty}^{\alpha+\beta x} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}z^2}\,dz, \tag{2}$$

and ϵ is a normal variate with zero mean and constant variance, independent of x. H, α and β are behaviour parameters; H, the maximum expected response, may be known *a priori*, but more frequently has to be estimated along with α and β. The method leads to a probit-analysis technique.

The hypothesis that $\text{var}\,(u) = \text{var}\,(\epsilon)$ is independent of x is not always strictly tenable, and Finney observed in his worked example of the repellent effect of lime-sulphur on the honey-bee that there was a strong indication that the variance of an observation increased with decreasing concentration of the repellent. A similar type of systematic heteroscedasticity is found in demand analysis in economics, where Aitchison and Brown (1954) have applied some of the methods of biological assay; here, a consumer's expenditure on a commodity is taken as the response u and his income, measured on a logarithmic scale, as the stimulus x; and there is strong evidence to support the hypothesis that

$$\text{var}\,(u) \propto \{E(u)\}^2, \tag{3}$$

that is, the coefficient of variation of the response is constant. This is well illustrated in Table 1, where the standard error and coefficient of variation of the *expenditure per person on meat* is given for ten groups of households classified by *income per person*. The corresponding values are also given for Finney's example; the results for this case are less conclusive.

2. THE ESTIMATION PROCEDURE

The model proposed is

$$u = HP(\alpha+\beta x)\,e^{\epsilon}, \tag{4}$$

where ϵ is normally distributed with zero mean and constant variance σ^2. Then

$$E(u) = H\,e^{\frac{1}{2}\sigma^2} P(\alpha+\beta x) \tag{5}$$

and

$$\text{var}\,(u) = (e^{\sigma^2}-1)\{E(u)\}^2 \propto \{E(u)\}^2, \tag{6}$$

so that the model satisfies hypothesis (3). There are often good *a priori* reasons for believing that multiplicative errors such as in the model (4) arise in practice.

Table 1. *Evidence of systematic heteroscedasticity*

Expenditures on meat*			Effect of lime-sulphur on the honey-bee†		
Household group	Standard error	Coefficient of variation	Treatment	Standard error‡	Coefficient of variation
1	4·6	0·36	*A*	3·2	0·69
2	5·7	·31	*B*	3·3	·43
3	6·5	·31	*C*	24·4	·97
4	8·6	·34	*D*	13·4	·38
5	9·1	·33	*E*	26·9	·43
6	8·9	0·30	*F*	29·2	0·42
7	11·0	·33	*G*	20·1	·29
8	10·9	·32	*H*	24·2	·27
9	12·5	·33	—	—	—
10	16·7	·41	—	—	—

* Data from the budgets of industrial working-class households collected by the Ministry of Labour in 1937–8.

† Data from Finney (1947, Table 33).

‡ These standard errors include a contribution from the row and column design of the experiment; the analysis of variance showed that this contribution is not significant.

If we write $v = \log u$ and $K = \log H$, then

$$v = K + \log P + \epsilon, \tag{7}$$

and the method of maximum likelihood leads to the minimization of the expression

$$\Sigma(v - K - \log P)^2 \tag{8}$$

with respect to K, α and β. There results a procedure formally similar to that given by Finney (in the second edition of his book); in his notation we have the iterative equations

$$bS_{xx} + \delta K S_{xx'} = S_{xy}, \tag{9}$$

$$bS_{xx'} + \delta K S_{x'x'} = S_{x'y}, \tag{10}$$

$$a = \bar{y} - b\bar{x} - \delta K \bar{x}', \tag{11}$$

where x' is still the auxiliary variable $x' = P/Z$, but the weighting factor is now $w = Z^2/P^2$ compared with Finney's Z^2. The working probit is given by the relation

$$y = Y + \frac{v - K - \log P}{Z/P}$$

$$= \left(Y - \frac{P}{Z}\log P\right) + (v - K)\frac{P}{Z}, \tag{12}$$

and

$$Y = a_0 + b_0 x \tag{13}$$

is calculated from the initial guesses at α and β, as are S_{xx}, $S_{xx'}$, etc. If a control group, for which $P = 1$, is used, $S_{x'x'}$ and $S_{x'y}$ must be increased by n_k and $n_k(k - K)$ respectively, where n_k is the number of subjects in the group and k is the mean of the logarithms of their responses. A separate estimate of σ^2 may be obtained from an analysis of variance.

The computations are considerably eased by the use of Table 2. This table gives for values of $Y = 1{\cdot}0\,(0{\cdot}1)\,9{\cdot}0$ the corresponding values of x', w and what may be considered the

minimum working probit, $Y_{\min.} = Y - (P/Z)\log P$, each to four places of decimals. The calculations were made by using values of P and Z from *Tables of the Probability Function*, vol. 2 (1942), published by the Federal Works Project Administration for the City of New York, and values of $-\log P$ from *The Probability Integral* (1939), vol. 7 of the *British Association Mathematical Tables*. The values of x' obtained were checked from the tables of Q/Z given in Finney (1947). The work of compiling the table has been carried out by Mr R. S. Bawa, to whom the authors wish to express their thanks.

Table 2. *Table of working probits*

Y	$Y_{\min.}$	$x'=P/Z$	$w=Z^2/P^2$	Y	$Y_{\min.}$	$x'=P/Z$	$w=Z^2/P^2$
1·0	3·4517	0·2367	17·8558	5·0	5·8687	1·2533	0·6366
1·1	3·5071	·2421	17·0599	5·1	5·9384	1·3599	·5407
1·2	3·5626	·2478	16·2837	5·2	6·0088	1·4813	·4557
1·3	3·6183	·2538	15·5272	5·3	6·0800	1·6202	·3810
1·4	3·6741	·2600	14·7903	5·4	6·1519	1·7797	·3157
1·5	3·7301	0·2666	14·0729	5·5	6·2246	1·9640	0·2592
1·6	3·7863	·2734	13·3751	5·6	7·2982	2·1780	·2108
1·7	3·8426	·2806	12·6969	5·7	6·3725	2·4276	·1697
1·8	3·8991	·2882	12·0381	5·8	6·4477	2·7206	·1351
1·9	3·9558	·2962	11·3987	5·9	6·5238	3·0665	·1063
2·0	4·0126	0·3046	10·7787	6·0	6·6007	3·4771	0·0827
2·1	4·0697	·3134	10·1781	6·1	6·6784	3·9675	·0635
2·2	4·1270	·3228	9·5968	6·2	6·7571	4·5571	·0482
2·3	4·1845	·3327	9·0347	6·3	6·8366	5·2705	·0360
2·4	4·2423	·3432	8·4917	6·4	6·9170	6·1394	·0265
2·5	4·3003	0·3543	7·9679	6·5	6·9982	7·2051	0·0193
2·6	4·3585	·3661	7·4631	6·6	7·0802	8·5214	·0138
2·7	4·4170	·3786	6·9772	6·7	7·1631	10·1589	·0097
2·8	4·4757	·3919	6·5101	6·8	7·2468	12·2111	·0067
2·9	4·5348	·4062	6·0618	6·9	7·3313	14·8026	·0046
3·0	4·5941	0·4214	5·6322	7·0	7·4165	18·100	0·0031
3·1	4·6538	·4376	5·2210	7·1	7·5025	22·330	·0020
3·2	4·7137	·4551	4·8282	7·2	7·5892	27·797	·0013
3·3	4·7741	·4739	4·4536	7·3	7·6765	34·923	·0008
3·4	4·8347	·4940	4·0971	7·4	7·7645	44·288	·0005
3·5	4·8958	0·5158	3·7585	7·5	7·8532	56·696	0·0003
3·6	4·9572	·5394	3·4375	7·6	7·9424	73·278	·0002
3·7	5·0190	·5649	3·1341	7·7	8·0321	95·628	·0001
3·8	5·0813	·5926	2·8478	7·8	8·1224	126·012	·0001
3·9	5·1440	·6227	2·5786	7·9	8·2132	167·682	·0000
4·0	5·2071	0·6557	2·3260	8·0	8·3044	225·33	—
4·1	5·2708	·6917	2·0899	8·1	8·3960	305·81	—
4·2	5·3349	·7313	1·8698	8·2	8·4881	419·16	—
4·3	5·3995	·7749	1·6654	8·3	8·5806	580·25	—
4·4	5·4648	·8230	1·4763	8·4	8·6734	811·27	—
4·5	5·5305	0·8764	1·3021	8·5	8·7665	1145·6	—
4·6	5·5969	·9357	1·1422	8·6	8·8600	1634·0	—
4·7	5·6639	1·0018	0·9963	8·7	8·9538	2353·9	—
4·8	5·7315	1·0759	·8638	8·8	9·0478	3425·0	—
4·9	5·7998	1·1593	·7441	8·9	9·1421	5033·6	—
5·0	5·8687	1·2533	0·6366	9·0	9·2366	7471·9	—

3. AN EXAMPLE

We shall illustrate the estimation procedure based on model (4) by applying it to the example considered by Finney, although for this case the hypothesis (3) is not well established; it allows, however, a convenient comparison with the method used by Finney, based on model (1).

A preliminary analysis of variance after a logarithmic transformation of the response yields an estimate s^2 of σ^2. This is given in Table 3 which compares with Table 35 of Finney.

Table 3. *Analysis of variance after logarithmic transformation*

	Degrees of freedom	Sum of squares	Mean square
Rows	7	3·193	
Columns	7	0·817	
Treatments	7	72·786	
Error	42	8·530	$s^2 = 0·203$
Total	63	85·326	

Table 4. *Computational scheme*

x	n	v	$v-K_0$	Y	y	x'	nw	nwx	nwx'	nwy
3	8	1·370	−3·012	3·4	3·347	0·4940	32·78	98·34	16·1933	109·7147
2	8	1·951	−2·431	3·9	3·630	0·6227	20·62	41·24	12·8401	74·8506
1	8	2·977	−1·405	4·4	4·309	0·8230	11·81	11·81	9·7196	50·8893
0	8	3·492	−0·890	4·9	4·768	1·1593	5·95	0·00	6·8978	28·3696
−1	8	4·077	−0·305	5·4	5·609	1·7797	2·53	−2·53	4·5026	14·1908
−2	8	4·128	−0·254	5·9	5·745	3·0665	0·848	−1·70	2·6004	4·8718
−3	8	4·167	−0·215	6·4	5·597	6·1394	0·216	−0·65	1·3261	1·2090
$-\infty$	8	4·475	+0·093	—	—	—	—	—	—	—
							74·76	146·51	54·0799	284·0958

$\bar{x} = 1·9597$, $\bar{x}' = 0·7234$, $\bar{y} = 3·8001$

$Snwx^2$	$Snwxx'$	$Snwx'^2$	$Snwxy$	$Snwx'y$	$Snwy^2$
397·1900	70·2980	56·1197	502·1732	223·1966	1107·8225
287·1212	105·9824	39·1203	556·7533	205·5093	1079·5937
110·0688	−35·6844	16·9994	−54·5801	17·6873	28·2288

$$n_k = \frac{8·0000}{24·9994}, \qquad n_k(k-K) = \frac{0·7440}{18·4313}.$$

We take as our initial guess at H, $H_0 = 80$, so that $K_0 = 4·382$; this gives a maximum expected response of approximately $H_0 e^{\frac{1}{2}s^2} = 88·5$; by the usual graphical determination of initial values of α and β we take $a_0 = 4·9$ and $b_0 = -0·5$, the same initial values as used by Finney.

The computational scheme is set out above in Table 4. Values of v, the mean of the logarithms of the response, are calculated at each concentration x and $v - K_0$ is recorded.

Y is easily obtained by (13); y is derived by adding the product of $v-K_0$ and $P/Z = x'$ to $Y_{\text{min.}}$; x' and w are recorded from Table 2 and the remainder of the computations completed.

The iterative equations (9), (10) and (11) are then

$$110{\cdot}0688b - 35{\cdot}6844\delta K_0 = -54{\cdot}5801,$$
$$-35{\cdot}6844b + 24{\cdot}9994\delta K_0 = \quad 18{\cdot}4313,$$
$$a = 3{\cdot}8001 - 1{\cdot}9597b - 0{\cdot}7234\delta K_0,$$

which give the values

$$K = 4{\cdot}437, \quad a = 4{\cdot}697, \quad b = -0{\cdot}478,$$

compared with initial values

$$K_0 = 4{\cdot}382, \quad a_0 = 4{\cdot}9, \quad b_0 = -0{\cdot}5.$$

The covariance matrix of b and K is given by

$$\sigma^2 \begin{bmatrix} S_{xx} & S_{xx'} \\ S_{xx'} & S_{x'x'} \end{bmatrix}^{-1}$$

and is estimated by inserting s^2 for σ^2 leading to

$$\begin{bmatrix} 0{\cdot}003433 & 0{\cdot}004900 \\ 0{\cdot}004900 & 0{\cdot}015115 \end{bmatrix}$$

Then $$|b - b_0| = 0{\cdot}022 < 0{\cdot}059 = \text{S.E.}(b)$$

and $$|K - K_0| = 0{\cdot}055 < 0{\cdot}123 = \text{S.E.}(K),$$

so that it is questionable whether a further iteration is necessary. A comparison of observed and expected responses leads to a mean square of 0·203, which, when compared with the error mean square 0·203, shows that the model fits satisfactorily. Two further iterations were actually carried out and the results of the three iterations are brought together in Table 5.

Table 5. *Results of successive iterations*

	K	a	b
Initial values	4·382	4·9	−0·5
First iteration	4·437	4·697	−0·478
Second iteration	4·436	4·715	−0·483
Third iteration	4·4355	4·7158	−0·4836

Quantile effective doses and their approximate confidence intervals may be calculated in the usual way; on the results of the third iteration log (ED 50) is found to be −0·588, and the corresponding 95% confidence interval is (−1·651, 0·125), compared with Finney's values of −0·256 and (−1·167, 0·356). It should also be remarked here that different estimates of the expected response are obtained by the two methods. For model (1), using Finney's estimates, we obtain for the estimate of $E(u)$

$$85{\cdot}59P(4{\cdot}875 - 0{\cdot}488x),$$

whereas with model (4) we estimate $E(u)$ by

$$e^{\frac{1}{2}s^2} HP(a + bx) = 90{\cdot}22P(4{\cdot}716 - 0{\cdot}484x).$$

4. GENERAL REMARKS

When the number of observations is not large the estimation procedure presented here involves little extra labour compared with that based on model (1). This extra work arises principally in the logarithmic transformation of the responses and, with the type of systematic heteroscedasticity (3), this is exactly the transformation (Cochran, 1938) which it is desirable to make before applying an analysis of variance technique. For a large number of observations the computations become more troublesome, though not prohibitive, and it would be desirable to investigate the effect of grouping on the estimates obtained. Also the method is directly applicable only where the responses are all positive or all negative; the occurrence of zero responses would require some modifications.

The authors wish to express their gratitude to Richard Stone, Director of the Department of Applied Economics, for his helpful criticism during the preparation of this note.

REFERENCES

AITCHISON, J. & BROWN, J. A. C. (1954). A synthesis of Engel curve theory. *Review of Economic Studies*, **21**. (In the Press.)

COCHRAN, W. G. (1938). Some difficulties in the statistical analysis of replicated experiments. *Emp. J. Exp. Agric.* **6**, 157.

FINNEY, D. J. (1947). *Probit Analysis: A Statistical Treatment of the Sigmoid Response Curve.* Cambridge University Press. 2nd ed. 1952.

THE USE OF THE HANKEL TRANSFORM IN STATISTICS. II. METHODS OF COMPUTATION

By R. D. LORD

Royal Technical College, Glasgow

1. The use of Fourier-Bessel series

1·1. In a previous paper (Lord, 1954, hereafter referred to as I), it was shown that when an s-dimensional random vector $\mathbf{X}$ has a spherical distribution with $P(r)\,dr$ for the probability of $r<|\mathbf{X}|<r+dr$, one can define a characteristic function

$$\Phi(\rho) = \Gamma(\tfrac{1}{2}s)\,2^{\frac{1}{2}s-1}\int_0^\infty (r\rho)^{-\frac{1}{2}s+1} J_{\frac{1}{2}s-1}(r\rho)\,P(r)\,dr, \tag{1}$$

with the associated inversion formula

$$P(r) = 2^{-\frac{1}{2}s+1}\{\Gamma(\tfrac{1}{2}s)\}^{-1}\int_0^\infty (r\rho)^{\frac{1}{2}s} J_{\frac{1}{2}s-1}(r\rho)\,\Phi(\rho)\,d\rho. \tag{2}$$

In problems of the addition of such vectors it will often be necessary to evaluate (2) approximately. The ordinary methods of numerical integration are difficult to apply, since the Bessel functions make the integrands oscillate irregularly between positive and negative values. However, when the probability is zero outside a certain range, $P(r)$ can be expanded in Fourier-Bessel series. This method has been employed by Bennett (1947) for Pearson's random-walk problem, but his treatment is not general and not based on the characteristic function.

If $s = 1$, the Fourier-Bessel series is an ordinary Fourier series. Suppose that $P(r) = 0$ if $r > R$. Then the characteristic function is

$$\phi(t) = \int_0^\infty P(r)\cos rt\,dr = \int_0^R P(r)\cos rt\,dr,$$

and so

$$\frac{2}{R}\int_0^R P(r)\cos(m\pi r/R)\,dr = \frac{2}{R}\phi(m\pi/R).$$

The left-hand expression is the coefficient of $\cos(m\pi r/R)$ in the expansion of $P(r)$ in a cosine series in the range $(0, R)$; for $m = 0$ the value given is to be halved. Hence

$$P(r) = \frac{1}{R} + \frac{2}{R}\sum_{m=1}^{\infty}\phi\left(\frac{m\pi}{R}\right)\cos\frac{m\pi r}{R}. \tag{3}$$

Integrating term-by-term

$$F(r) = \frac{r}{R} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\phi\left(\frac{m\pi}{R}\right)\sin\frac{m\pi r}{R}, \tag{4}$$

where $F(r)$ is the distribution function, i.e. the probability of $0 \leqslant |\mathbf{X}| \leqslant r$.

For general values of s we proceed as follows. A function $f(r)$ can be expanded in the interval $(0, R)$ in the series

$$f(r) = \sum_{m=1}^{\infty}\alpha_m J_\nu(j_m r/R),$$

where $j_1, j_2, \ldots$ are the zeros of $J_\nu(x)$ and

$$\alpha_m = [2/\{J^2_{\nu+1}(j_m)\,R^2\}]\int_0^R rf(r)\,J_\nu(j_m r/R)\,dr.$$

Taking $f(r) = r^{-\frac{1}{2}s}P(r)$, $\nu = \frac{1}{2}s - 1$ and $j_1, j_2, \ldots$ as the zeros of $J_{\frac{1}{2}s-1}(x)$ we have

$$P(r) = r^{\frac{1}{2}s}\sum_{m=1}^{\infty}\alpha_m J_{\frac{1}{2}s-1}(j_m r/R), \tag{5}$$

where

$$\alpha_m = [2/\{J^2_{\frac{1}{2}s}(j_m)\,R^2\}]\int_0^R r^{1-\frac{1}{2}s}P(r)\,J_{\frac{1}{2}s-1}(j_m r/R)\,dr \tag{6a}$$

$$= \frac{2}{\Gamma(\frac{1}{2}s)\,R^2 J^2_{\frac{1}{2}s}(j_m)}\left(\frac{j_m}{2R}\right)^{\frac{1}{2}s-1}\Phi\left(\frac{j_m}{R}\right), \tag{6}$$

the integral over $(0, R)$ being replaceable by an integral over $(0, \infty)$, since $P(r)$ is zero for $r > R$. Integrating, we find

$$F(r) = \left(\frac{r}{R}\right)^{\frac{1}{2}s}\sum_{m=1}^{\infty}\beta_m J_{\frac{1}{2}s}(j_m r/R), \tag{7}$$

where

$$\beta_m = \frac{1}{\Gamma(\frac{1}{2}s)\,J^2_{\frac{1}{2}s}(j_m)}\,(\tfrac{1}{2}j_m)^{\frac{1}{2}s-2}\,\Phi\left(\frac{j_m}{R}\right). \tag{8}$$

1·2. If $s = 2$

$$P(r) = \frac{2r}{R^2}\sum_{m=1}^{\infty}\frac{1}{J^2_1(j_m)}\,\Phi\left(\frac{j_m}{R}\right)J_0\left(\frac{j_m r}{R}\right), \tag{9}$$

$$F(r) = \frac{2r}{R}\sum_{m=1}^{\infty}\frac{1}{j_m J^2_1(j_m)}\,\Phi\left(\frac{j_m}{R}\right)J_1\left(\frac{j_m r}{R}\right), \tag{10}$$

where $j_1, j_2, \ldots$ are the zeros of $J_0(x)$. Tables of $J_0(j_m x)$ by Goodwin & Staton (1948) are available to ease computation of $P(r)$.

If $s = 3$

$$J_{\frac{1}{2}s-1}(r) = J_{\frac{1}{2}}(r) = (2/\pi r)^{\frac{1}{2}}\sin r, \quad J_{\frac{1}{2}s}(r) = J_{\frac{3}{2}}(r) = (2/\pi r)^{\frac{1}{2}}\left(\frac{\sin r}{r} - \cos r\right),$$

so that $j_m = m\pi$ and $J^2_{\frac{1}{2}s}(m) = 2/(m\pi^2)$. Hence

$$P(r) = \frac{2\pi r}{R^2}\sum_{m=1}^{\infty} m\,\Phi\left(\frac{m\pi}{R}\right)\sin\frac{m\pi r}{R}, \tag{11}$$

$$F(r) = \frac{2r}{R}\sum_{m=1}^{\infty}\Phi\left(\frac{m\pi}{R}\right)\left(\frac{R}{m\pi r}\sin\frac{m\pi r}{R} - \cos\frac{m\pi r}{R}\right). \tag{12}$$

1·3. This is perhaps the most accurate method in general of computing $F(r)$, but the convergence can be very slow, since it is determined almost entirely by the speed with which $\Phi(\rho)$ decreases. Occasionally convergence can be slightly improved by a preliminary integration by parts. In general, we can integrate (2) N times by parts and then expand $f(r) = r^{N-\frac{1}{2}s}P(r)$ in a Fourier-Bessel series. The resulting formula for $P(r)$ is then integrated to give $F(r)$, but the general result is simple only in the case $s = 2$, $N = 1$. The method is not usually suitable for $P(r)$, since it means finding a small quantity as the sum of terms which are not so small.

1·4. When $P(r)$ does not vanish outside a finite range the method may still be usable. We consider the case $s \geqslant 2$. Expand $P(r)$ over the range $(0, R)$. The error in using (6) for the coefficients arises from replacing the finite integral in (6a) by an infinite integral and is proportional to

$$\epsilon_m = \int_R^\infty r^{1-\frac{1}{2}s} P(r) J_{\frac{1}{2}s-1}(j_m r/R)\, dr.$$

If we assume that $P(r)$ decreases for $r > R$, the integrand, which starts from zero at $r = R$ and alternates in sign, will have a graph consisting of a series of half-waves of diminishing amplitude and nearly equal lengths. Hence $|\epsilon_m|$ is less than the area under the first half-wave, in which $P(r)$ has its maximum at $r = R$ and $|r^{1-\frac{1}{2}s} J_{\frac{1}{2}s-1}(j_m r/R)|$ has its maximum at $r = Rk_m/j_m$, where k_m is the mth zero of $J_{\frac{1}{2}s}(x)$. Hence

$$|\epsilon_m| < \frac{j_{m+1}-j_m}{j_m} RP(R) \left(\frac{k_m R}{j_m}\right)^{1-\frac{1}{2}s} |J_{\frac{1}{2}s-1}(k_m)|,$$

and for the error $\delta\beta_m$ in β_m we have

$$|\delta\beta_m| < 2RP(R) \frac{j_{m+1}-j_m}{j_m^2 J_{\frac{1}{2}s}^2(j_m)} \left(\frac{j_m}{k_m}\right)^{\frac{1}{2}s-1} |J_{\frac{1}{2}s-1}(k_m)|.$$

Now (Watson, 1944, p. 200) as x tends to infinity

$$J_{\frac{1}{2}s-1}^2(x) + J_{\frac{1}{2}s}^2(x) \sim 2/(\pi x),$$

so that as m tends to infinity both $j_m J_{\frac{1}{2}s}^2(j_m)$ and $k_m J_{\frac{1}{2}s-1}^2(k_m)$ tend to $2/\pi$. Also $(j_{m+1}-j_m)$ tends to π. Little error for our purpose is caused by replacing the above expressions by their limits and j_m/k_m by 1. Hence

$$|\delta\beta_m| < \{2\pi^3/(k_m j_m^2)\}^{\frac{1}{2}} RP(R),$$

and δF, the error in $F(r)$ satisfies

$$\begin{aligned} |\delta F| &< (r/R)^{\frac{1}{2}s} \sum_{m=1}^{\infty} |\delta\beta_m| \\ &< A(r/R)^{\frac{1}{2}s} RP(R), \end{aligned}$$

where

$$A = (2\pi^3)^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{1}{j_m k_m^{\frac{1}{2}}}.$$

A depends on s, and adequate rough estimates are not difficult to obtain. For $s = 2$ we find $A < 2{\cdot}8$ so

$$|\delta F| < 2{\cdot}8rP(R). \tag{13}$$

A closer estimate could of course be made by examining the terms of the series (7) more closely but is not worth while when the estimate is used in the following way. Use the best estimate of $P(r)$ to find an R for which the estimated error is substantially less than the error one is willing to make and use this value of R in (7). One estimate of $P(r)$ is always available, namely, the normal approximation I (36) or the modifications to be given later, but it may not be very close and that is why a margin of error is left.

1·5. As a test $F(r)$ for the sum of two coplanar vectors, each with an exponential distribution with parameter b, was considered. By I (48)

$$F(r) = 1 - \frac{1}{8}\left(\frac{r}{b}\right)^3 K_3\left(\frac{r}{b}\right)$$

exactly, and by I (43) $\Phi(\rho) = (1+b^2\rho^2)^{-3}$.

Hence, by I (18), the second moment is

$$\mu_2 = 12b^2.$$

The normal distribution with $\mu_2 = 12b^2$ was therefore used to estimate $P(r)$. For the sum of two vectors the normal estimate is very inaccurate. However, it was judged that with this estimate four-decimal accuracy in $F(r)$ should almost certainly be obtained by choosing R so that (13) gave $|\delta F| < 10^{-7}$. $R = 15b$ was chosen as a convenient value satisfying this requirement.

The series (7) was then used to calculate $F(r)$ for the values $r = 1{\cdot}5b$, $3{\cdot}0b$, $6{\cdot}0b$, $13{\cdot}5b$, i.e. for $r = 0{\cdot}1R$, $0{\cdot}2R$, $0{\cdot}4R$, $0{\cdot}9R$. The errors were found to be $+60$, -2, $+5$, -15 units in the sixth decimal place, the true values being 0·226364, 0·587675, 0·933259, 0·999804. These errors include the effects of truncating the series (at about the 30th term in each case). The trend of the omitted terms showed that the truncation error for $r = 0{\cdot}1R$ was rather more than 30 units in the sixth decimal place, but almost negligible in the other cases. Thus the series gave four-decimal accuracy in the worst case and something substantially better in the other cases.

The convergence was slow, partly on account of the large value of R and partly on account of $\Phi(\rho)$ being merely the square of the characteristic function of a single vector. For the sum of three vectors about 18 terms would have given the same accuracy. The results of this test can, however, be considered satisfactory.

2. Expansions in terms of Laguerre polynomials

2·1. We now find error terms in the normal approximation by expanding $P(r)$ and $F(r)$ in series whose first terms are the normal approximations. From the definitions of the cumulants λ_{2n} [I (19)],

$$\begin{aligned}\Phi(\rho) &= \exp\left\{-\frac{\lambda_2\rho^2}{2s}\right\}\exp\left\{\frac{\lambda_4}{2.4s(s+2)}\rho^4 - \frac{\lambda_6}{2.4.6s(s+2)(s+4)}\rho^6 + \ldots\right\}\\ &= \exp\left\{-\frac{\lambda_2\rho^2}{2s}\right\}\left\{1 + \frac{A_4}{2.4s(s+2)}\rho^4 - \frac{A_6}{2.4.6s(s+2)(s+4)}\rho^6 + \ldots\right\},\end{aligned} \tag{14}$$

say, on expanding the second exponential in powers of ρ. But this expansion differs from the passage I (19) back to I (18) only in the absence of $\mu_2 = \lambda_2$ and in replacing the other μ's by A's. Thus putting $\lambda_2 = 0$ and writing A_4, A_6, ... for λ_4, λ_6, ... in I (21) we have

$$\left.\begin{aligned} A_4 = \lambda_4,\quad A_6 = \lambda_6,\quad A_8 &= \lambda_8 + 3\frac{(s+4)(s+6)}{s(s+2)}\lambda_4^2,\\ A_{10} &= \lambda_{10} + 10\frac{(s+6)(s+8)}{s(s+2)}\lambda_6\lambda_4.\end{aligned}\right\} \tag{15}$$

Further values can be found from Kendall (1943, equation (3·30)) by replacing his μ_ν, κ_ν by $A_{2\nu}/\{2.4\ldots 2\nu.s(s+2)\ldots(s+2\nu-2)\}$ and $\lambda_{2n}/\{2.4\ldots 2\nu.s(s+2)\ldots(s+2\nu-2)\}$ respectively.

In terms of moments

$$A_{2N} = \sum_{\nu=0}^{N}(-)^\nu\binom{N}{\nu}\left(\frac{\mu_2}{s}\right)^\nu (s+2N-2)(s+2N-4)\ldots(s+2N-2\nu)\mu_{2N-2\nu}.$$

This follows by applying Maclaurin's theorem to expand $\Phi(\rho)\exp(\frac{1}{2}\lambda_2\rho^2/s)$.

Substituting from (14) into (2)

$$P(r) = \psi_0(r) + A_4\psi_4(r) - A_6\psi_6(r) + \ldots, \tag{16}$$

where $\psi_0(r)$ has characteristic function $\exp(-\frac{1}{2}\lambda_2\rho^2/s)$ and is therefore the normal probability function

$$\psi_0(r) = \frac{2}{\Gamma(\frac{1}{2}s)}\left(\frac{s}{2\lambda_2}\right)^{\frac{1}{2}s} r^{s-1}\exp\left(-\frac{sr^2}{2\lambda_2}\right), \tag{17}$$

while if $N > 0$, using Watson (p. 394 (3)),

$$\left.\begin{aligned}\psi_{2N}(r) &= \frac{1}{2.4\ldots 2Ns(s+2)\ldots(s+2N-2)}\frac{r^{\frac{1}{2}s}}{2^{\frac{1}{2}s-1}\Gamma(\frac{1}{2}s)}\int_0^\infty e^{-\frac{1}{2}\lambda_2\rho^2/s}\rho^{\frac{1}{2}s+N}J_{\frac{1}{2}s-1}(r\rho)\,d\rho \\ &= \frac{2}{N!\,\Gamma(\frac{1}{2}s)}\left(\frac{s}{2\lambda_2}\right)^{N+\frac{1}{2}s} r^{s-1}e^{-\frac{1}{2}sr^2/\lambda_2}\,{}_1F_1\left(-N, \tfrac{1}{2}s; \frac{sr^2}{2\lambda_2}\right) \\ &= \frac{2}{\Gamma(\frac{1}{2}s+N)}\left(\frac{s}{2\lambda_2}\right)^{N+\frac{1}{2}s} r^{s-1}e^{-\frac{1}{2}sr^2/\lambda_2}\,L_N^{(\frac{1}{2}s+1)}\left(\frac{sr^2}{2\lambda_2}\right),\end{aligned}\right\} \tag{18}$$

i.e.

$$P(r)/\psi_0(r) = 1 + \sum_{N=2}^{\infty}(-)^N\frac{A_{2N}}{N!}\left(\frac{s}{2\lambda_2}\right)^N {}_1F_1\left(-N, \tfrac{1}{2}s; \frac{sr^2}{2\lambda_2}\right). \tag{19}$$

Here ${}_1F_1(\alpha, \gamma; x)$ is the confluent hypergeometric function

$${}_1F_1 = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots,$$

while $L_m^{(\alpha)}(x)$ is the generalized Laguerre polynomial

$$L_m^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{m!}\left(\frac{d}{dx}\right)^m (e^x x^{m+\alpha}) = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+m)}{m!}\,{}_1F_1(-m, 1+\alpha; x).$$

To find the distribution function we need the value of

$$\Psi_{2N}(r) = \int_0^r \psi_{2N}(t)\,dt.$$

Using the Laguerre form, and putting $u = \frac{1}{2}st^2/\lambda_2$, $U = \frac{1}{2}sr^2/\lambda_2$, we obtain, for $N > 0$,

$$\left.\begin{aligned}\Psi_{2N}(r) &= \frac{1}{\Gamma(\frac{1}{2}s+N)}\left(\frac{s}{2\lambda_2}\right)^N \int_0^U u^{\frac{1}{2}s-1}e^{-u}L_N^{(\frac{1}{2}s-1)}(u)\,du \\ &= \frac{1}{N\Gamma(\frac{1}{2}s+N)}\left(\frac{s}{2\lambda_2}\right)^N \int_0^U \frac{d}{du}[e^{-u}u^{\frac{1}{2}s}L_{N-1}^{(\frac{1}{2}s)}(u)\,du]\,du \\ &= \frac{1}{N\Gamma(\frac{1}{2}s+N)}\left(\frac{s}{2\lambda_2}\right)^N e^{-U}U^{\frac{1}{2}s}L_{N-1}^{(\frac{1}{2}s)}(U) \\ &= \frac{1}{N!\,\Gamma(\frac{1}{2}s)\lambda_2}\left(\frac{s}{2\lambda_2}\right)^{N-1} e^{-U}U^{\frac{1}{2}s}\,{}_1F_1(-N+1, \tfrac{1}{2}s+1; U).\end{aligned}\right\} \tag{20}$$

The distribution function is then given by

$$F(r) = \Psi_0(r) + A_4\Psi_4(r) - A_6\Psi_6(r) + \ldots. \tag{21}$$

2·2. For $s = 1$, the expansion is equivalent to the well-known Type A expansion in terms of Hermite polynomials. It is equivalent for the plane random-walk problem to Pearson's expansion in terms of what he calls ω-functions, and for the three-dimensional one to an

expansion given by Rayleigh. When the distribution is that of the sum of n vectors with identical distributions (cumulants $\lambda_2, \lambda_4, \ldots$) the first few terms are as follows.

For $s = 2$, denoting $r^2/(n\lambda_2)$ by u,

$$P(r) = \frac{2}{n\lambda_2} r\, e^{-u}\Big[1 + \frac{\lambda_4}{2n\lambda_2^2}(1 - 2u + \tfrac{1}{2}u^2) - \frac{\lambda_6}{6n^2\lambda_2^3}(1 - 3u + \tfrac{3}{2}u^2 + \tfrac{1}{6}u^3) + \frac{\lambda_8 + 18n\lambda_4}{24n^3\lambda_2^4}(1 - 4u + 3u^2 - \tfrac{2}{3}u^3 + \tfrac{1}{24}u^4) - \ldots\Big], \quad (22)$$

$$F(r) = 1 - e^{-u}\Big[1 - \frac{\lambda_4}{2n\lambda_2^2}u(1 - \tfrac{1}{2}u) + \frac{\lambda_6}{6n^2\lambda_2^3}u(1 - u + \tfrac{1}{8}u^2) - \frac{\lambda_8 + 18n\lambda_4}{24n^3\lambda_2^4}u(1 - \tfrac{3}{2}u + \tfrac{1}{2}u^2 - \tfrac{1}{24}u^3) + \ldots\Big]. \quad (23)$$

For $s = 3$, denoting $3r^2/(2n\lambda_2)$ by U,

$$P(r) = \Big(\frac{24}{n\pi\lambda_2}\Big)^{\frac{1}{2}} U\, e^{-U}\Big[1 + \frac{9\lambda_4}{8n\lambda_2^2}(1 - \tfrac{4}{3}U + \tfrac{4}{15}U^2) - \frac{9\lambda_6}{16n^2\lambda_2^3}(1 - 2U + \tfrac{4}{3}U^2 - \tfrac{8}{105}U^3) + \frac{27}{1280}\frac{5\lambda_8 + 63n\lambda_4^2}{n^3\lambda_2^4}(1 - \tfrac{8}{3}U + \tfrac{8}{5}U^2 - \tfrac{32}{105}U^3 + \tfrac{8}{945}U^4) - \ldots\Big] \quad (24)$$

$$F(r) = \alpha\Big(r\sqrt{\frac{3}{n\lambda_2}}\Big) - \frac{2}{\sqrt{\pi}}U^{\frac{1}{2}}e^{-U}\Big[1 - \frac{3\lambda_4}{4n\lambda_2^2}U(1 - \tfrac{2}{5}U) + \frac{3\lambda_6}{8n^2\lambda_2^3}U(1 - \tfrac{4}{5}U + \tfrac{4}{35}U^2) - \frac{9}{320}\frac{5\lambda_8 + 63n\lambda_4^2}{n^3\lambda_2^4}(1 - \tfrac{6}{5}U + \tfrac{12}{35}U^2 - \tfrac{8}{315}U^3) + \ldots\Big], \quad (25)$$

where $$\alpha(x) = \sqrt{(2/\pi)}\int_0^x e^{-\frac{1}{2}t^2}\,dt.$$

2·3. These formulae have been very little tested for $s > 1$, but they would appear to behave rather like the Type A series ($s = 1$) and to give satisfactory approximations for nearly normal distributions except at the tails. In the case of the distribution of the sum of n coplanar random vectors of equal magnitude, Pearson concluded that five terms of the series were enough to give four-decimal accuracy for $n \geqslant 7$, but investigations which the author hopes to publish shortly suggest that he was rather optimistic.

The following are the results of a few tests on (23) applied to the distribution considered by Pearson; $F_n(r)$ is the distribution function for the sum of n vectors of length a. Take first $r = a$, since $F_n(a) = (n+1)^{-1}$ exactly. Five terms of (23) gave the following errors in units of the fifth decimal place, the corresponding error in the normal approximation being given in brackets:

$n = 4$: $-618(+2120)$, $n = 6$: $-74(+1067)$,
$n = 8$: $-26(+650)$, $n = 10$: $-11(+425)$.

Next $r = na$, since $F_n(na) = 1$ exactly. The errors were now

$n = 4$: $-2584(-1832)$, $n = 6$: $-501(-248)$,
$n = 8$: $-95(-34)$, $n = 10$: $-17(-4{\cdot}5)$.

It is a reasonable conclusion that the approximation is fairly satisfactory except near the tail, where the normal approximation can be better.

2·4. We can find formulae for approximate normalization (i.e. formulae for r', where r' is to make $\Psi_0(r') = F(r)$ as accurately as possible) by comparing (21) with the expansion of $\Psi_0(r')$ in powers of $(r'-r)$. The results rapidly become very complicated and we content ourselves with

$$r'-r = s\lambda_4 r/(8\lambda_2^2), \tag{26}$$

$$r'-r = \frac{s\lambda_4}{8\lambda_2^2}r - \frac{s\lambda_4}{8(s+2)\lambda_2^3}r^3 - \frac{s^2\lambda_6}{48\lambda_2^3}r + \frac{s^2(s-1)\lambda_4^2}{128\lambda_2^4}r. \tag{27}$$

For the sum of n vectors with the same distribution, the right-hand side of (26) is the only term of order n^{-1}, while that of (27) includes all terms of order n^{-1} and n^{-2}.

These formulae have not been adequately tested but it appears that (26) improves the normal approximation substantially for moderate values of r and that the first two terms of (27) carry the improvement nearer the tail of the distribution.

In general, it would appear that for greatest accuracy it would be best to use (19) to estimate $P(r)$ and then employ the methods of § 1.

REFERENCES

Bennett, W. R. (1947). Distribution of the sum of randomly phased components. *Quart. Appl. Math.* **5**, 385–93.

Goodwin, E. J. & Staton, J. (1948). Table of $J_0(j_{0,s}r)$. *Quart. J. Mech.* **1**, 220–4.

Kendall, M. G. (1943). *The Advanced Theory of Statistics*, **1**. London.

Lord, R. D. (1954). The use of the Hankel transform in statistics. I. General theory and examples. *Biometrika*, **41**, 44–55.

Pearson, K. (1906). Mathematical contributions to the theory of evolution. XV. A mathematical theory of random migration. *Drap. Co. Res. Mem. Biom. Ser.*

Watson, G. N. (1944). *A Treatise on the Theory of Bessel Functions*, 2nd ed. (or 1st ed. 1922). Cambridge University Press.

A REDUCTION FORMULA FOR NORMAL MULTIVARIATE INTEGRALS

BY R. L. PLACKETT
University of Liverpool

1. We suppose that the vector $\mathbf{x}$ of n random variables with zero means and unit variances has a non-singular normal multivariate distribution with probability density function

$$\phi_n(x_1, x_2, \ldots, x_n; c_{11}, c_{12}, \ldots, c_{nn}) = (2\pi)^{-\frac{1}{2}n} |\mathbf{C}|^{\frac{1}{2}} \exp(-\tfrac{1}{2}\mathbf{x}'\mathbf{C}\mathbf{x}). \quad (1)$$

Here $\mathbf{C}$ is the inverse of the variance matrix of $\mathbf{x}$, which is denoted by $\mathbf{R}$, and has elements $\{\rho_{ij}\}$. The integral

$$\Phi_n(a_1, a_2, \ldots, a_n; c_{11}, c_{12}, \ldots, c_{nn}) = \int_{a_1}^{\infty}\int_{a_2}^{\infty}\ldots\int_{a_n}^{\infty} \phi_n \, dx_1 dx_2 \ldots dx_n \quad (2)$$

has been extensively tabulated for n equal to 1 and 2, but a simple calculation will show that comparable tables are scarcely feasible for higher values of n. However, the joint distribution of several continuous dependent variables is often assumed to conform to multivariate normality, and experiments may arise in which we wish to compare observed and expected frequencies, as, for example, in testing models for the joint action of several poisons (Plackett & Hewlett, 1952).

When $a_1, a_2, \ldots, a_n$ are all zero, we denote the value of Φ_n by Φ_n^0; it can be interpreted as the area of a general simplex constructed on the surface of a sphere in n dimensions. Schläfli (1858) found a differential recurrence relation for Φ_n^0, and showed that, when n is odd, Φ_n^0 can be expressed as a combination of values of $\Phi_{n-1}^0, \Phi_{n-3}^0, \ldots, \Phi_2^0$. For particular values of $\mathbf{R}$, he also gave some exact expressions and several identities. Coxeter (1935) has established further properties of Φ_4^0 when the correlation coefficients ρ_{13}, ρ_{14} and ρ_{24} are all zero. When all the correlation coefficients are equal to ρ, Ruben (1954) has shown that Schläfli's results yield a simple formula connecting Φ_n^0 with Φ_{n-2}^0; and he gives, in his Table 1, the values of Φ_n^0 for

$$1/\rho = 2, 3, \ldots, 12$$

and

$$n = 0, 1, 2, \ldots, 51 - 1/\rho.$$

Another general approach to the problem is by means of the tetrachoric series, the details of which when n is 2 are given by Kendall (1945). This series has been generalized by Kendall (1941) and Moran (1948), who state explicit formulae for Φ_n when n is 3 and 4; but although the tetrachoric series will always converge, it does so very slowly when the absolute values of the correlation coefficients are near unity, and some alternative approach then appears desirable, as David (1953) has indicated.

A direct attack is occasionally rewarding. Thus, when

$$c_{pq} = \begin{cases} 2, & p = q, \\ -1, & p - q = \pm 1, \\ 0, & \text{otherwise}, \end{cases}$$

Anis & Lloyd (1953) prove that Φ_n^0 is $1/(n+1)$.

In what follows, we obtain a reduction formula for Φ_n, which permits us to express Φ_3 and Φ_4 as finite sums of single integrals of tabulated functions. These integrals must in general be evaluated by numerical quadrature, but under certain conditions we shall derive simple approximations to them. Schläfli's differential recurrence formula can be deduced from ours by putting

$$a_1 = a_2 = \ldots = a_n = 0,$$

and interpreting the result geometrically.

2. Our starting point is the system of partial differential equations represented by

$$\partial\phi_n/\partial\rho_{ij} = \partial^2\phi_n/\partial x_i \partial x_j. \tag{3}$$

The equation for n equal to 2 has been known for some time. To establish (3) for all n, we write ϕ_n as the transform of its characteristic function:

$$\phi_n = (2\pi)^{-n} \int\int \ldots \int \exp(-i\mathbf{t}'\mathbf{x} - \tfrac{1}{2}\mathbf{t}'\mathbf{R}\mathbf{t})\, dt_1 dt_2 \ldots dt_n.$$

Both sides of (3) then become

$$-(2\pi)^{-n} \int\int \ldots \int t_i t_j \exp(-i\mathbf{t}'\mathbf{x} - \tfrac{1}{2}\mathbf{t}'\mathbf{R}\mathbf{t})\, dt_1 dt_2 \ldots dt_n.$$

We next differentiate Φ_n with respect to all $\{\rho_{ij}\}$:

$$\begin{aligned}\frac{\partial\Phi_n}{\partial\rho_{12}} &= \int_{a_1}^{\infty}\int_{a_2}^{\infty}\ldots\int_{a_n}^{\infty} (\partial\phi_n/\partial\rho_{12})\, dx_1 dx_2 \ldots dx_n \\ &= \int_{a_3}^{\infty}\int_{a_4}^{\infty}\ldots\int_{a_n}^{\infty}\left\{\int_{a_1}^{\infty}\int_{a_2}^{\infty} (\partial^2\phi_n/\partial x_1 \partial x_2)\, dx_1 dx_2\right\} dx_3 dx_4 \ldots dx_n \\ &= \int_{a_3}^{\infty}\int_{a_4}^{\infty}\ldots\int_{a_n}^{\infty} \phi_n(a_1, a_2, x_3, x_4, \ldots, x_n;\, c_{11}, c_{12}, \ldots, c_{nn})\, dx_3 dx_4 \ldots dx_n. \end{aligned} \tag{4}$$

To bring this into a more convenient form, we note that the conditional distributions of the normal multivariate distribution are expressible in terms of deviations from regression. In fact, let

$$\mathbf{x}_1' = (x_1, x_2, \ldots, x_m),$$

$$\mathbf{x}_2' = (x_{m+1}, x_{m+2}, \ldots, x_n),$$

and

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \cdots & \cdots \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix},$$

where $\mathbf{C}_{11}$ is $m \times m$. If the inverse of $\mathbf{C}$, correspondingly partitioned, is

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \cdots & \cdots \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix},$$

then

$$\mathbf{R}_{11}^{-1} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21},$$

$$|\mathbf{R}_{11}| = |\mathbf{C}_{22}|/|\mathbf{C}|,$$

and

$$\mathbf{R}_{21}\mathbf{R}_{11}^{-1} = -\mathbf{C}_{22}^{-1}\mathbf{C}_{21}.$$

Partitioning $\mathbf{x}$ and $\mathbf{C}$ in (1), and using the last three equations, we get

$$\begin{aligned}\phi_n = {} & (2\pi)^{-\frac{1}{2}m} |\mathbf{R}_{11}|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\mathbf{x}_1'\mathbf{R}_{11}^{-1}\mathbf{x}_1\} \\ & \times (2\pi)^{-\frac{1}{2}(n-m)} |\mathbf{C}_{22}|^{\frac{1}{2}} \exp\{-\tfrac{1}{2}(\mathbf{x}_2 - \mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{x}_1)'\, \mathbf{C}_{22}(\mathbf{x}_2 - \mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{x}_1)\}. \end{aligned} \tag{5}$$

The special case when m is 2 is relevant here. Put

$$b_r = \{(\rho_{1r}-\rho_{2r}\rho_{12})a_1+(\rho_{2r}-\rho_{1r}\rho_{12})a_2\}/(1-\rho_{12}^2) \quad (r = 3, 4, \ldots, n).$$

Equation (4) becomes

$$\partial\Phi_n/\partial\rho_{12} = (2\pi)^{-1}(1-\rho_{12}^2)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}(a_1^2-2\rho_{12}a_1a_2+a_2^2)/(1-\rho_{12}^2)\}$$
$$\times\Phi_{n-2}(a_3-b_3, a_4-b_4, \ldots, a_n-b_n; c_{33}, c_{34}, \ldots, c_{nn}). \qquad (6)$$

Similar expressions for the other first-order partial derivatives can be written down at once.

3. For convenience in terminology, we consider any variance matrix $\mathbf{R}$ as defining a point P with co-ordinates $\{\rho_{ij}\}$ in a space of $\binom{n}{2}$ dimensions. When $a_1, a_2, \ldots, a_n$ are fixed, the value of Φ_n depends solely on the position of P and will be denoted by $\Phi_n(P)$. Suppose that K, with co-ordinates $\{\kappa_{ij}\}$, is any point where the value of Φ_n is obvious or can be calculated directly from the tables of Φ_1 and Φ_2. Let $\{\lambda_{ij}(t)\}$ be the co-ordinates of the point L which divides KP in the ratio $t:1-t$. Then

$$\lambda_{ij}(t) = t\rho_{ij}+(1-t)\kappa_{ij}.$$

Write $\mathbf{R}(t)$ for the matrix with unit diagonal elements and non-diagonal elements $\{\lambda_{ij}(t)\}$; thus $\mathbf{R}(0)$ corresponds to the point K and $\mathbf{R}(1)$ to P. Since

$$\mathbf{R}(t) = t\mathbf{R}(1)+(1-t)\mathbf{R}(0),$$

it is positive definite for $0<t\leqslant 1$. Denote the inverse of $\mathbf{R}(t)$ by $\mathbf{C}(t)$, with elements $c_{pq}(t)$. $\mathbf{C}(t)$ is positive definite for the same range of t, and therefore the type of argument leading to equation (6) can be applied at any intermediate point on KP, so that $\partial\Phi_n/\partial\lambda_{12}$ at the point L is obtained from (6) on replacing ρ_{ij} by $\lambda_{ij}(t)$ and c_{pq} by $c_{pq}(t)$.

To derive the basic reduction formula, we integrate the differential element $d\Phi_n$ along the line KP, and obtain

$$\Phi_n(P) = \Phi_n(K)+\sum_{i<j}\int_{\kappa_{ij}}^{\rho_{ij}}\frac{\partial\Phi_n}{\partial\lambda_{ij}}(L)\,d\lambda_{ij}. \qquad (7)$$

On coming to the numerical evaluation of (7), we shall change the variable of integration by writing

$$\lambda_{ij} = \cos\theta_{ij}.$$

This has the effect of removing $(1-\lambda_{ij}^2)^{-\frac{1}{2}}$ from the integrand, thus shortening the computations, and avoiding the singularity which occurs at the lower limit when κ_{ij} is ± 1.

4. There is considerable flexibility in the choice of K, but two general rules can be suggested.

(i) Select a point near to P, although, as will appear when we give an example with four variables, the length of the path of integration in the space of the correlation coefficients is, by itself, no guide to the amount of care which we have to bestow on the process of numerical integration.

(ii) Choose $\mathbf{R}(0)$ to be a matrix of rank $(n-1)$ with all but one of its correlation coefficients the same as those of $\mathbf{R}(1)$; the second term on the right of (7) then consists of a single integral. To prove that this is always possible, let $\mathbf{R}_{11}$ be the first $(n-1)$ rows and columns of $\mathbf{R}$, and partition the remainder of $\mathbf{R}$ as before. Denote by $\mathbf{R}_{12}^*$ the vector obtained from $\mathbf{R}_{12}$ on replacing $\rho_{n-1,n}$ by $\rho_{n-1,n}^*$. Since

$$|\mathbf{R}| = |\mathbf{R}_{11}|(1-\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{R}_{12}),$$

we have to solve

$$\mathbf{R}^*_{21}\mathbf{R}^{-1}_{11}\mathbf{R}^*_{12} = 1$$

for $\rho^*_{n-1,n}$. On rearrangement,

$$(\mathbf{R}^*_{21}-\mathbf{R}_{21})\,\mathbf{R}^{-1}_{11}(\mathbf{R}^*_{12}-\mathbf{R}_{12}) + 2(\mathbf{R}^*_{21}-\mathbf{R}_{21})\,\mathbf{R}^{-1}_{11}\mathbf{R}_{12} - (1-\mathbf{R}_{21}\mathbf{R}^{-1}_{11}\mathbf{R}_{12}) = 0,$$

which is a quadratic equation with positive discriminant. It therefore provides two real values of $\rho^*_{n-1,n}$ and either of these is the correlation coefficient of x_{n-1} and

$$x^*_n = \psi_1 x_1 + \psi_2 x_2 + \ldots + \psi_{n-1}x_{n-1},$$

where $\mathbf{R}_{11}\psi = \mathbf{R}^*_{12}$.

When n is 3,

$$\rho^*_{23} = \rho_{12}\rho_{13} \pm \{(1-\rho^2_{12})\,(1-\rho^2_{13})\}^{\frac{1}{2}} \tag{8}$$

and

$$x^*_3 = \{(\rho_{13}-\rho_{12}\rho^*_{23})\,x_1 + (\rho^*_{23}-\rho_{12}\rho_{13})\,x_2\}/(1-\rho^2_{12}). \tag{9}$$

The tables of Φ_1 and Φ_2 suffice to calculate

$$\Phi_3(K) = \Pr(x_1 > a_1, x_2 > a_2, x^*_3 > a_3).$$

When n is 4,

$$\begin{aligned}(1-\rho^2_{12})\,\rho^*_{34} &= \rho_{14}(\rho_{13}-\rho_{12}\rho_{23}) + \rho_{24}(\rho_{23}-\rho_{12}\rho_{13}) \\ &\pm \{(1-\rho^2_{12}-\rho^2_{13}-\rho^2_{23}+2\rho_{12}\rho_{13}\rho_{23})\,(1-\rho^2_{12}-\rho^2_{14}-\rho^2_{24}+2\rho_{12}\rho_{14}\rho_{24})\}^{\frac{1}{2}}. \end{aligned} \tag{10}$$

The coefficients in x^*_4 are obtained by solving the system of equations

$$\begin{aligned}\psi_1 \;+\rho_{12}\psi_2+\rho_{13}\psi_3 &= \rho_{14}, \\ \rho_{12}\psi_1+\;\psi_2\;+\rho_{23}\psi_3 &= \rho_{24}, \\ \rho_{13}\psi_1+\rho_{23}\psi_2+\;\psi_3\;\; &= \rho^*_{34},\end{aligned}$$

and

and are checked by evaluating

$$\rho_{14}\psi_1+\rho_{24}\psi_2+\rho^*_{34}\psi_3 = 1.$$

The combination of probabilities forming $\Phi_4(K)$ can be derived mechanically as follows. In the space S of points (x_1, x_2, x_3) let S_1, S_2, S_3 and T be the regions defined by $x_1 > a_1$, $x_2 > a_2$, $x_3 > a_3$ and $x^*_4 > a_4$ respectively. There will always exist some null relation of the form

$$\bar{S}_1 S_2 \bar{S}_3 T = 0,$$

where a bar signifies the complementary set. On expanding

$$(S-S_1)\,S_2(S-S_3)\,T = 0,$$

we get

$$S_1S_2S_3T = S_1S_2T + S_2S_3T - S_2T.$$

$\Phi_4(K)$ therefore involves at most four trivariate normal integrals.

5. Our methods lead to the following expression for the trivariate normal integral:

$$\begin{aligned}\Phi_3(P) &= \Pr(x_1 > a_1, x_2 > a_2, x^*_3 > a_3) \\ &+ (2\pi)^{-1}\int_{\cos^{-1}\rho_{13}}^{\cos^{-1}\rho^*_{23}} \exp\{-(a_2^2+a_3^2-2a_2a_3\cos\theta)/2\sin^2\theta\} \\ &\times \Phi_1\left\{\frac{(a_1-\rho_{12}a_2-\rho_{13}a_3)+(\rho_{13}a_2+\rho_{12}a_3)\cos\theta - a_1\cos^2\theta}{\sin\theta(1-\rho^2_{12}-\rho^2_{13}+2\rho_{12}\rho_{13}\cos\theta-\cos^2\theta)^{\frac{1}{2}}};\, 1\right\} d\theta. \end{aligned} \tag{11}$$

Suppose, for example, that

$$a_1 = -1{\cdot}2, \quad a_2 = -1{\cdot}0, \quad a_3 = 0{\cdot}5$$

and

$$\mathbf{R} = \begin{bmatrix} 1 & 0{\cdot}7 & 0{\cdot}2 \\ & 1 & -0{\cdot}4 \\ & & 1 \end{bmatrix}.$$

Equation (8) gives

$$\rho_{23}^* = 0{\cdot}14 \pm 0{\cdot}699714.$$

Taking the critical value to be $-0{\cdot}559714$, and using (9),

$$x_3^* = 1{\cdot}160392x_1 - 1{\cdot}371989x_2.$$

The inequalities $x_1 < -1{\cdot}2$ and $x_2 > -1{\cdot}0$ together imply $x_3^* < -0{\cdot}020481$, and so the set $\bar{S}_1 S_2 T$ is null. Hence

$$S_1 S_2 T = S_2 T,$$

giving $$\Phi_3(K) = \Pr(x_2 > -1{\cdot}0, x_3^* > 0{\cdot}5;\ \rho_{23}^* = -0{\cdot}559714) = 0{\cdot}587191.$$

Since ρ_{23} and ρ_{23}^* are negative here, we replace θ by $\pi - \theta$ in (11). The second stage of the work then consists in the evaluation of

$$(2\pi)^{-1} \int_{\cos^{-1}(0{\cdot}559714)}^{\cos^{-1}(0{\cdot}4)} \exp\{-(0{\cdot}625 - 0{\cdot}5\cos\theta)/\sin^2\theta\} \times \Phi_1\left\{\frac{-0{\cdot}6 - 0{\cdot}15\cos\theta + 1{\cdot}2\cos^2\theta}{\sin\theta(0{\cdot}47 - 0{\cdot}28\cos\theta - \cos^2\theta)^{\frac{1}{2}}};\ 1\right\} d\theta.$$

To achieve six significant figures in $\Phi_3(P)$, we carry six as far as the final product for the integrand, where we step down to five, thus minimizing rounding-off errors. The integrand has a maximum near the lower limit and the contribution from this section was estimated by the three-eighths rule, using strips of $\frac{1}{40}$th of the interval. Thereafter, the integrand decreases steadily, but the first differences have a maximum, so that Gregory's formula was applied twice, firstly over nine strips of $\frac{1}{40}$th, and secondly over seven strips of $\frac{1}{10}$th. The final result is 0·016343, whence

$$\Phi_3(P) = 0{\cdot}603534.$$

Of course, less work is required if less accuracy will suffice. In our example, $\Phi_3(P)$ is given with an error of only one unit in the fourth decimal place merely by applying the trapezoidal rule to the integral.

6. When $a_1, a_2, \ldots$ are all zero, equation (7) simplifies considerably, and we are easily led to the known results for n equal to 2 and 3. Taking K at the origin,

$$\Phi_2^0(P) = \tfrac{1}{4} + (2\pi)^{-1} \int_{\cos^{-1}\rho_{12}}^{\frac{1}{2}\pi} d\theta = \{\cos^{-1}(-\rho_{12})\}/2\pi;$$

and

$$\Phi_3^0(P) = \tfrac{1}{8} + (4\pi)^{-1}\left\{\int_{\cos^{-1}\rho_{12}}^{\frac{1}{2}\pi} d\theta + \int_{\cos^{-1}\rho_{13}}^{\frac{1}{2}\pi} d\theta + \int_{\cos^{-1}\rho_{23}}^{\frac{1}{2}\pi} d\theta\right\}$$
$$= \{\cos^{-1}(-\rho_{12}) + \cos^{-1}(-\rho_{13}) + \cos^{-1}(-\rho_{23}) - \pi\}/4\pi.$$

When n is 4, put

$$\sigma_{pq}(t) = c_{pq}(t)/\{c_{pp}(t)\, c_{qq}(t)\}^{\frac{1}{2}},$$

and (7) becomes

$$\Phi_4^0(P) = \Phi_4^0(K) + (1/4\pi^2) \sum_{p<q} \int_{\kappa_{ij}}^{\rho_{ij}} \cos^{-1}\sigma_{pq}(1 - \lambda_{ij}^2)^{-\frac{1}{2}}\, d\lambda_{ij}, \tag{12}$$

where (p, q, i, j) is a permutation of $(1, 2, 3, 4)$.

Two approximations follow. If all the $\{\rho_{ij}\}$ are small, we take K at the origin and get

$$\sigma_{ij} \simeq -\tfrac{1}{4}\rho_{ij}.$$

The integrals in (12) now group themselves in pairs to give

$$\Phi_4^0(P) \simeq \{\cos^{-1}(-\rho_{12})\cos^{-1}(-\rho_{34}) + \cos^{-1}(-\rho_{13})\cos^{-1}(-\rho_{24}) + \cos^{-1}(-\rho_{14})\cos^{-1}(-\rho_{23})\}/4\pi^2 - \tfrac{1}{8}. \quad (13)$$

If we expand (13), ignoring quantities of order ρ^3, we get the first three terms of Moran's formula (3).

When all the differences $\{\rho_{ij} - \kappa_{ij}\}$ are small, and provided that $\sigma_{pq}(0)$ is not close to ± 1, we can write

$$\cos^{-1}\sigma_{pq}(t) \simeq t\cos^{-1}\sigma_{pq}(1) + (1-t)\cos^{-1}\sigma_{pq}(0) = \alpha_{pq} + \beta_{pq}\lambda_{ij}(t).$$

Under these conditions

$$\int_{\kappa_{ij}}^{\rho_{ij}} \cos^{-1}\sigma_{pq}(1-\lambda_{ij}^2)^{-\frac{1}{2}}\,d\lambda_{ij} \simeq \alpha_{pq}(\cos^{-1}\kappa_{ij} - \cos^{-1}\rho_{ij}) + \beta_{pq}\{(1-\kappa_{ij}^2)^{\frac{1}{2}} - (1-\rho_{ij}^2)^{\frac{1}{2}}\}. \quad (14)$$

This approximation cannot be used when $\mathbf{R}(0)$ is a matrix of rank 3, for the equation

$$(1-\sigma_{pq}^2) = (1-\lambda_{ij}^2)\,|\,\mathbf{R}(t)\,| \Big/ \begin{vmatrix} 1 & \lambda_{pi} & \lambda_{pj} \\ & 1 & \lambda_{ij} \\ & & 1 \end{vmatrix} \begin{vmatrix} 1 & \lambda_{qi} & \lambda_{qj} \\ & 1 & \lambda_{ij} \\ & & 1 \end{vmatrix}$$

then implies that

$$\sigma_{pq}(0) = \pm 1.$$

A brief table of Φ_4^0 is given in an appendix, in order to extend the range of (14) by providing additional starting-points K.

7. We shall illustrate the calculation of Φ_4^0 by an example in which all the correlation coefficients are fairly large in absolute magnitude. Suppose that

$$\mathbf{R} = \begin{bmatrix} 1 & -0{\cdot}60 & 0{\cdot}85 & 0{\cdot}75 \\ & 1 & -0{\cdot}70 & -0{\cdot}80 \\ & & 1 & 0{\cdot}65 \\ & & & 1 \end{bmatrix}.$$

Equation (10) gives $0{\cdot}64\rho_{34}^* = 0{\cdot}4745 \pm 0{\cdot}14928580.$

Taking the positive sign $\rho_{34}^* = 0{\cdot}97466531$

and $x_4^* = -0{\cdot}287x_1 - 0{\cdot}234x_2 + 1{\cdot}055x_3.$

Since $S_1S_2\bar{S}_3T$ is null,

$$\Phi_4^0(K) = \Pr(x_1 > 0, x_2 > 0, x_4^* > 0;\ \rho_{12} = -0{\cdot}60, \rho_{14} = 0{\cdot}75, \rho_{24} = -0{\cdot}80)$$
$$= 0{\cdot}067487.$$

Turning to the second term in (12),

$$\sigma_{12} = -\begin{vmatrix} -0{\cdot}60 & 0{\cdot}85 & 0{\cdot}75 \\ -0{\cdot}70 & 1 & \cos\theta \\ -0{\cdot}80 & \cos\theta & 1 \end{vmatrix}^{-\frac{1}{2}} \begin{vmatrix} 1 & -0{\cdot}70 & -0{\cdot}80 \\ -0{\cdot}70 & 1 & \cos\theta \\ -0{\cdot}80 & \cos\theta & 1 \end{vmatrix}^{-\frac{1}{2}} \begin{vmatrix} 1 & 0{\cdot}85 & 0{\cdot}75 \\ 0{\cdot}85 & 1 & \cos\theta \\ 0{\cdot}75 & \cos\theta & 1 \end{vmatrix}^{-\frac{1}{2}},$$

and we thus have the problem of computing

$$-(1/4\pi^2)\int_{\cos^{-1}(0\cdot97\ldots)}^{\cos^{-1}(0\cdot65)} \cos^{-1}\left\{\frac{-0\cdot595+1\cdot205\cos\theta-0\cdot6\cos^2\theta}{(-0\cdot13+1\cdot12\cos\theta-\cos^2\theta)^{\frac{1}{2}}(-0\cdot285+1\cdot275\cos\theta-\cos^2\theta)^{\frac{1}{2}}}\right\}d\theta.$$

This integral presents two features which make the computations more laborious than in the previous example. In the first place, $\cos\theta$ must be recorded to eight decimal places in order to obtain six in the argument of the inverse cosine. Secondly, this argument is 1 at the lower limit, and because

$$\cos^{-1}(1-x) = \sqrt{(2x)}\,(1+x/12+3x^2/160+5x^3/896+\ldots)$$

the integrand behaves like $x^{\frac{1}{2}}$ when x is close to zero, so that the usual methods of numerical integration break down. The difficulty is overcome by using a formula given by Jeffreys & Jeffreys (1946), § 9·092, namely

$$\int_0^{3h}(\alpha x^{\frac{1}{2}}+\beta x^{\frac{3}{2}}+\gamma x^{\frac{5}{2}})\,dx = h(6\sqrt{3}y_1/7+12\sqrt{6}y_2/35+16y_3/35).$$

This was applied with strips of $\frac{1}{250}$th of the interval, and followed by Gregory's formula over seven strips of $\frac{1}{250}$th, eight strips of $\frac{1}{50}$th and eight strips of $\frac{1}{10}$th. The final result is 0·025164, whence

$$\Phi_4^0(P) = 0\cdot042323.$$

To test the adequacy of (14), we now work through the same example taking

$$\mathbf{R}(t) = \begin{bmatrix} 1 & -1+0\cdot40t & 1-0\cdot15t & 1-0\cdot25t \\ & 1 & -1+0\cdot30t & -1+0\cdot20t \\ & & 1 & 1-0\cdot35t \\ & & & 1 \end{bmatrix}.$$

$\mathbf{R}(0)$ is a matrix of rank 1. It corresponds to the system of relations

$$x_1 = -x_2 = x_3 = x_4,$$

from which we infer that

$$\Phi_4^0(K) = 0.$$

The quantities $\{\sigma_{pq}(t)\}$ are computed from the adjoint of $\mathbf{R}(t)$, namely,

$$\mathfrak{R}(t) = t^2\begin{bmatrix} 0\cdot2175-0\cdot042t & -0\cdot0775+0\cdot01225t & -0\cdot1675+0\cdot037t & -0\cdot1275+0\cdot0285t \\ & 0\cdot1475-0\cdot02625t & 0\cdot1025-0\cdot02375t & 0\cdot1225-0\cdot02775t \\ & & 0\cdot1975-0\cdot04t & 0\cdot0725-0\cdot014t \\ & & & 0\cdot1775-0\cdot036t \end{bmatrix}.$$

As each row of $\mathfrak{R}(t)$ is completed, it is checked by forming its product—theoretically zero—with some other row of $\mathbf{R}(t)$. Of the six integrals to be evaluated, a typical one is

$$I_{14} = -0\cdot25\int_0^1 \{1-(1-0\cdot25t)^2\}^{-\frac{1}{2}}\cos^{-1}\left\{\frac{0\cdot1025-0\cdot02375t}{(0\cdot1475-0\cdot02625t)^{\frac{1}{2}}(0\cdot1975-0\cdot04t)^{\frac{1}{2}}}\right\}dt$$

$$= -\int_0^{\cos^{-1}(0\cdot75)} \cos^{-1}\left\{\frac{0\cdot0075+0\cdot095\cos\theta}{(0\cdot0425+0\cdot105\cos\theta)^{\frac{1}{2}}(0\cdot0375+0\cdot160\cos\theta)^{\frac{1}{2}}}\right\}d\theta.$$

The values taken by the inverse cosine at the limits of integration are 0·92662 and 0·96446; and

$$0\cdot92662\,(1-t)+0\cdot96446t = 1\cdot07798-0\cdot15136(1-0\cdot25t).$$

Hence, using (14),

$$I_{14} \simeq -1{\cdot}07798\cos^{-1}(0{\cdot}75) + 0{\cdot}15136\,(1-0{\cdot}75^2)^{\frac{1}{2}} = -0{\cdot}67898.$$

On the other hand, applying Gregory with five strips to the second formulation,

$$I_{14} = -0{\cdot}67827.$$

Although the length of the path of integration in the space of the $\{\rho_{ij}\}$ is double what it was previously—0·705 against 0·325—and although the number of integrations has been increased from one to six, yet the total amount of work in evaluating $\Phi_4^0(P)$ to six decimal places is roughly the same as before. The exact and approximate results for all six integrals are collected in the table below. Approximation (14) therefore leads to an estimate of $\Phi_4^0(P)$ differing from the exact value by less than one unit in the fourth decimal place; and it thus appears to work quite satisfactorily, considering that the differences $\{\rho_{ij}-\kappa_{ij}\}$ are not particularly small.

Integral	Approximation (14)	Numerical integration
I_{12}	1·08628	1·08643
I_{13}	−0·40438	−0·40371
I_{14}	−0·67898	−0·67827
I_{23}	1·80380	1·80439
I_{24}	1·60801	1·60864
I_{34}	−1·74700	−1·74661
Total	1·66773	1·67087
$\div 4\pi^2$	0·042244	0·042324

8. The calculation of Φ_4 by our method is similar to, but more laborious than, the calculation of Φ_3. Once again, single integrals will suffice. If suggestion (ii) is accepted, $\Phi_4(K)$ can be expressed as a trivariate normal integral and is evaluated as described above; the integrand of the remaining term is a product of probabilities and ordinates of the normal bivariate distribution. For Φ_5 and Φ_6, double integrals are required, and, in general, the effect of repeatedly applying equation (7) is to reduce the multiplicity of a normal integral by more than half. Whether our results offer a practical method of approach when n exceeds 4 is a moot point, but they are perhaps worth considering when values of Φ_5, Φ_6, ... are urgently needed.

I am indebted to a referee for simplifying my original proof of equation (3).

APPENDIX

Our method of calculating Φ_n depends on knowing the values of this function for certain values of **R**. Such knowledge need not be systematic, however—as would be essential if we were interpolating—and consequently the values of Φ_4^0 given below, although forming a rather haphazard collection, may nevertheless be useful. Since the results of Anis & Lloyd (1953) and Ruben (1954) are available in recent issues of this journal, they are not reproduced here.

(A)
$$\mathbf{R} = \begin{bmatrix} 1 & \rho_{12} & 0 & 0 \\ & 1 & \rho_{23} & 0 \\ & & 1 & \rho_{34} \\ & & & 1 \end{bmatrix}.$$

Put $\rho_{12} = -\cos\alpha, \quad \rho_{23} = -\cos\beta, \quad \rho_{34} = -\cos\gamma,$

where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi, \quad 0 \leqslant \beta \leqslant \pi, \quad 0 \leqslant \gamma \leqslant \frac{1}{2}\pi.$

The relationship between the function $f(\alpha, \beta, \gamma)$ of Schläfli, the infinite sum $S(\alpha, \beta, \gamma)$ of Coxeter, and Φ_4^0, is expressed by

$$\Phi_4^0 = f(\alpha, \beta, \gamma)/16 = S(\tfrac{1}{2}\pi - \alpha, \beta, \tfrac{1}{2}\pi - \gamma)/8\pi^2.$$

The following table was computed from a set of exact values of $f(\alpha, \beta, \gamma)$ given by Schläfli (third paper, p. 98); the first ten are also derived by Coxeter:

$-\rho_{12}$	$-\rho_{23}$	$-\rho_{34}$	Φ_4^0
0·500000	0·500000	0·500000	0·008333
·707107	·500000	·500000	·002604
·500000	·707107	·500000	·000868
·809017	·500000	·500000	·000069
·309017	·500000	·500000	·013264
·809017	·500000	·309017	·001389
·309017	·809017	·500000	·000278
·500000	·309017	·809017	·005278
·809017	·309017	·809017	·000417
·309017	·809017	·309017	·004583
0·500000	0·500000	0·790569	0·000408
·500000	·790569	·250000	·001631
·790569	·250000	·790569	·002446
·500000	·309017	·925615	·000217
·309017	·925615	·135045	·000930
·925615	·135045	·809017	·001624
·135045	·809017	·500000	·002301

From each value of Φ_4^0, five others can be generated by changing the signs of the variables x_1, x_2, x_3 and x_4. For example,

$$\Phi_4^0 \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ & 1 & -\frac{1}{2} & 0 \\ & & 1 & -\frac{1}{2} \\ & & & 1 \end{bmatrix} = \Phi_3^0 \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{1}{2} \\ & & 1 \end{bmatrix} - \Phi_4^0 \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ & 1 & -\frac{1}{2} & 0 \\ & & 1 & -\frac{1}{2} \\ & & & 1 \end{bmatrix}.$$

Thus, the set of six associated with the first tabular entry is as follows:

Formation	ρ_{12}	ρ_{23}	ρ_{24}	Φ_4^0
	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{120}$
$-x_1$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{30}$
$-x_1, -x_2$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{120}$
$-x_2$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{40}$
$-x_1, -x_4$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{11}{120}$
$-x_1, -x_3$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{15}$

(B) Two more values of Φ_4^0 are obtainable by elementary combinatorial methods, based on a set of uncorrelated random variables $y_1, y_2, \ldots, y_5$, all of which have the same distribution and variance $\frac{1}{2}$. For example, the first **R** is the variance matrix of y_3-y_1, y_4-y_1, y_3-y_2, y_4-y_2; and, out of 4! ordered arrangements of y_1, y_2, y_3, y_4, there are $(2!)^2$ which ensure that all four differences are positive. Hence Φ_4^0 is $\frac{1}{6}$ whether or not anything further is specified about the distribution of x_1, x_2, x_3 and x_4.

Formation	**R**	Φ_4^0
13 14 23 24	$\begin{matrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ & 1 & 0 & \frac{1}{2} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{matrix}$	$\frac{1}{6}$
12 13 15 45	$\begin{matrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ & 1 & \frac{1}{2} & 0 \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{matrix}$	$\frac{3}{10}$

The first **R** belongs to a set of four; and the second, to a set of six.

REFERENCES

Anis, A. A. & Lloyd, E. H. (1953). On the range of partial sums of a finite number of independent normal variates. *Biometrika*, **40**, 35–42.

Coxeter, H. S. M. (1935). The functions of Schläfli and Lobatschefsky. *Quart. J. Math.* **6**, 13–29.

David, F. N. (1953). A note on the evaluation of the multivariate normal integral. *Biometrika*, **40**, 458–9.

Jeffreys, H. & Jeffreys, B. S. (1946). *Methods of Mathematical Physics*. Cambridge University Press.

Kendall, M. G. (1941). Proof of relations connected with the tetrachoric series and its generalization. *Biometrika*, **32**, 196–8.

Kendall, M. G. (1945). *The Advanced Theory of Statistics*, **1**. London: Griffin.

Moran, P. A. P. (1948). Rank correlation and product-moment correlation. *Biometrika*, **35**, 203–6.

Plackett, R. L. & Hewlett, P. S. (1952). Quantal responses to mixtures of poisons. *J.R. Statist. Soc.* ser. B, **14**, 141–54.

Ruben, H. (1954). On the moments of order statistics in samples from normal populations. *Biometrika*, **41**, 200–27.

Schläfli, L. (1858). On the multiple integral $\int^n dx dy \ldots dz$ whose limits are

$$p_1 = a_1 x + b_1 y + \ldots + h_1 z > 0, \quad p_2 > 0, \ldots, p_n > 0 \quad \text{and} \quad x^2 + y^2 + \ldots + z^2 < 1.$$

Quart. J. Pure Appl. Math. **2**, 269–301; **3**, 54–68, 97–108.

ON THE COMPARISON OF TWO MEANS: FURTHER DISCUSSION OF ITERATIVE METHODS FOR CALCULATING TABLES

By W. H. TRICKETT and B. L. WELCH
Leeds College of Technology and University of Leeds

1. Introduction

One of the present authors (Welch, 1947) has given a solution of the problem of comparing two means, which has formed the basis of tables calculated by Alice A. Aspin (1949). This solution was presented in a somewhat condensed form, and has been criticized on the ground that it is not easy to follow. It is therefore proposed in the present paper to give a more extended treatment of the problem in the hope that certain points, at least, will be clarified.

We do not intend to proceed immediately to the original solution (which was given as a series development), but to discuss first a numerical iterative method which possibly demonstrates better what is being attempted. The connexion of this numerical method with the series will then be shown.

2. Formulation of problem

It is assumed that n_1 individuals are available from a normal population with mean $\bar{X}_1$ and standard deviation σ_1; and n_2 individuals from a second normal population with mean $\bar{X}_2$ and standard deviation σ_2. The corresponding sample statistics are $\bar{x}_1$, s_1, $\bar{x}_2$ and s_2. We wish to make probability statements about the difference, $(\bar{X}_1 - \bar{X}_2)$, between the two population means.

It is well known that $(\bar{x}_1 - \bar{x}_2)$ is distributed normally about $(\bar{X}_1 - \bar{X}_2)$ with variance $(\sigma_1^2/n_1 + \sigma_2^2/n_2)$. If σ_1 and σ_2 were known, therefore, it would be possible to refer the ratio

$$u = \frac{(\bar{x}_1 - \bar{x}_2) - (\bar{X}_1 - \bar{X}_2)}{\sqrt{(\sigma_1^2/n_1 + \sigma_2^2/n_2)}} \tag{1}$$

to the standard normal tables.

If the σ's are not known it is natural to consider instead the ratio

$$v = \frac{(\bar{x}_1 - \bar{x}_2) - (\bar{X}_1 - \bar{X}_2)}{\sqrt{(s_1^2/n_1 + s_1^2/n_2)}}, \tag{2}$$

where now the true sampling standard deviation of $(\bar{x}_1 - \bar{x}_2)$ is replaced by an estimated value. (Nothing is being assumed about the ratio σ_1/σ_2, so that σ_1^2 and σ_2^2 are estimated separately by s_1^2 and s_2^2.) It is known that the ratio $M_1 = s_1^2/\sigma_1^2$ has the standard mean-square distribution

$$p(M)\,dM = \frac{1}{\Gamma(\frac{1}{2}f)} e^{-\frac{1}{2}fM} \left(\frac{fM}{2}\right)^{\frac{1}{2}f-1} d\left(\frac{fM}{2}\right), \tag{3}$$

where $f = (n_1 - 1) = f_1$ (say), while $M_2 = s_2^2/\sigma_2^2$ is distributed similarly with $f = (n_2 - 1) = f_2$ (say). Also these mean-square distributions are independent of the distribution of u. Using these facts, the probability that v is less than some specified value v_0 may then be deduced as follows.

From (1) and (2) we have $$v = u\{M_1\gamma + M_2(1-\gamma)\}^{-1}, \tag{4}$$

where $$\gamma = \frac{(\sigma_1^2/n_1)}{(\sigma_1^2/n_1 + \sigma_2^2/n_2)}. \tag{5}$$

Hence, for *given* s_1^2 and s_2^2 (i.e. for *fixed* M_1 and M_2) we have

$$\Pr(v < v_0) = N\{\sqrt{[(M_1\gamma + M_2\overline{1-\gamma})]}\, v_0\}, \tag{6}$$

where $$N\{x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}u^2}\, du. \tag{7}$$

Thus for *unrestricted* s_1^2 and s_2^2 it follows that

$$\Pr(v < v_0) = \int_0^\infty \int_0^\infty N\{\sqrt{(M_1\gamma + M_2\overline{1-\gamma})}\, v_0\}\, p(M_1)\, p(M_2)\, dM_1\, dM_2. \tag{8}$$

By writing $$b = \frac{f_1 M_1}{(f_1 M_1 + f_2 M_2)}, \quad w = (f_1 M_1 + f_2 M_2), \tag{9}$$

this may be transformed to

$$\Pr(v < v_0) = \int_{b=0}^{1} \int_{w=0}^{\infty} N\left\{\sqrt{\left(\frac{\gamma b}{f_1} + \frac{(1-\gamma)(1-b)}{f_2}\right)} v_0 \sqrt{w}\right\} p(w)\, p(b)\, dw\, db, \tag{10}$$

where $$p(w)\, dw = \frac{1}{\Gamma\left(\frac{f_1+f_2}{2}\right)} e^{-\frac{1}{2}w} \left(\frac{w}{2}\right)^{\frac{1}{2}(f_1+f_2)-1} d\left(\frac{w}{2}\right), \tag{11}$$

and $$p(b)\, db = \frac{1}{B(\frac{1}{2}f_1, \frac{1}{2}f_2)} b^{\frac{1}{2}f_1 - 1}(1-b)^{\frac{1}{2}f_2 - 1}\, db. \tag{12}$$

On replacing N by the explicit expression (7) it is found that we can integrate (10) out for w, leaving

$$\Pr(v < v_0) = \int_{b=0}^{1} T_{(f_1+f_2)}\left\{\sqrt{(f_1+f_2)} \sqrt{\left(\frac{\gamma b}{f_1} + \frac{(1-\gamma)(1-b)}{f_2}\right)} v_0\right\} p(b)\, db, \tag{13}$$

where $$T_F\{x\} = \int_{-\infty}^{x} \frac{1}{B(\frac{1}{2}, \frac{1}{2}F)} \frac{dt/\sqrt{F}}{(1 + t^2/F)^{\frac{1}{2}(F+1)}}. \tag{14}$$

In the transition from (8) to (13) the normal probability integral, N, has been replaced by the function $T_{(f_1+f_2)}$, which is the probability integral of the Student distribution with (f_1+f_2) degrees of freedom. Since good tables of the Student integral exist (Pearson & Hartley, 1954, Table 9), the evaluation of $\Pr(v < v_0)$ is therefore reduced by (13) to a single quadrature. (It does not seem, in general, possible to simplify further than this, although in the particular cases $\gamma = 0$ and $\gamma = 1$, the integral may be shown to reduce to $T_{f_2}\{v_0\}$ and $T_{f_1}\{v_0\}$ respectively. These cases correspond to $\sigma_1 = 0$ and $\sigma_2 = 0$, and from direct considerations it can be seen that v is then actually distributed in the Student distribution.)

It is seen from (13) that $\Pr(v < v_0)$ depends only on the population parameters through the quantity γ, which from (5) is seen to depend only on the ratio of σ_1 and σ_2. The value, v_0, such that $\Pr(v < v_0)$ has some specified value P, say, is therefore a function of γ. Let this function be $\phi(\gamma)$. (It is understood, of course, that ϕ involves f_1, f_2, and P as well as γ, but it is not necessary to write this down explicitly.) Provided then that $\phi(\gamma)$ is chosen to satisfy the equation

$$P = \int_{b=0}^{1} T_{(f_1+f_2)}\left\{\sqrt{(f_1+f_2)} \sqrt{\left(\frac{\gamma b}{f_1} + \frac{(1-\gamma)(1-b)}{f_2}\right)} \phi(\gamma)\right\} p(b)\, db, \tag{15}$$

we shall have $$\Pr\{v < \phi(\gamma)\} = P. \tag{16}$$

Given a table of $\phi(\gamma)$, obtained by solving (15) for a number of different γ, we would be able in theory to look up the P-significance level of $\{(\bar{x}_1-\bar{x}_2)-(\bar{X}_1-\bar{X}_2)\}/\sqrt{(s_1^2/n_1+s_2^2/n_2)}$ in any particular problem and hence make an inference about $(\bar{X}_1-\bar{X}_2)$. This, however, is only in theory for, in general, we shall not know γ and hence will not be able to enter the table. As an approximation we might estimate γ from the sample by means of the quantity

$$c=\frac{s_1^2/n_1}{(s_1^2/n_1+s_2^2/n_2)}, \tag{17}$$

and then adopt $\phi(c)$ as our critical value of v. If this is done, however, we shall not have $\Pr\{v<\phi(c)\}$ now *exactly* equal to P. For $\phi(c)$ is itself a random variable, varying about $\phi(\gamma)$.

If one wishes to make a statement which shall have a chance exactly equal to P of being right, it seems that $\phi(c)$ will have to be replaced by a slightly different function of c—say $v(c)$. We seek then to find a function $v(c)$ such that

$$\Pr\{v<v(c)\}=P \tag{18}$$

exactly, whatever γ may be. One may be optimistic in hoping such a function to exist. Whether a solution exists, or not, however, let us assume for the moment that one does and see how one might set about calculating it.

From (17) we have
$$c=\frac{M_1\gamma}{(M_1\gamma+M_2\overline{1-\gamma})}, \tag{19}$$

and hence from (4)

$$\Pr\{v<v(c)\}=\Pr\left\{u<\sqrt{(M_1\gamma+M_2\overline{1-\gamma})}\,v\left(\frac{M_1\gamma}{M_1\gamma+M_2(1-\gamma)}\right)\right\}. \tag{20}$$

For *given* M_1 and M_2 we have, therefore,

$$\Pr\{v<v(c)\}=N\left\{\sqrt{(M_1\gamma+M_2\overline{1-\gamma})}\,v\left(\frac{M_1\gamma}{M_1\gamma+M_2\overline{1-\gamma}}\right)\right\}. \tag{21}$$

Hence for *unrestricted* M_1 and M_2,

$$\Pr\{v<v(c)\}=\int_0^\infty\int_0^\infty N\left\{\sqrt{(M_1\gamma+M_2\overline{1-\gamma})}\,v\left(\frac{M_1\gamma}{M_1\gamma+M_2\overline{1-\gamma}}\right)\right\}p(M_1)\,p(M_2)\,dM_1\,dM_2. \tag{22}$$

On making the transformation (9) this now leads to

$$\Pr\{v<v(c)\}=\int_{b=0}^1 T_{(f_1+f_2)}\left\{\sqrt{(f_1+f_2)}\sqrt{\left(\frac{\gamma b}{f_1}+\frac{(1-\gamma)(1-b)}{f_2}\right)}\,v\left(\frac{\gamma b/f_1}{\gamma b/f_1+(1-\gamma)(1-b)/f_2}\right)\right\}p(b)\,db. \tag{23}$$

The expression on the right of (23) is to equal P for all γ. Thus we have to solve the integral equation

$$P=\int_{b=0}^1 T_{(f_1+f_2)}\left\{\sqrt{(f_1+f_2)}\sqrt{\left(\frac{\gamma b}{f_1}+\frac{(1-\gamma)(1-b)}{f_2}\right)}\,v(c)\right\}p(b)\,db, \tag{24}$$

where in the expression $v(c)$ it is understood that

$$c=\frac{\gamma b/f_1}{\gamma b/f_1+(1-\gamma)(1-b)/f_2}. \tag{25}$$

The unknown in the equation is the functional form $v(c)$ which appears non-linearly. It is not to be expected, therefore, that a numerical solution will be easy to obtain.

3. An iterative method of solution

Suppose that from independent considerations a first approximation—$v_0(c)$, say—to the solution of (22) is known, and that we write

$$v(c) = v_0(c) + v_1(c). \tag{26}$$

Then, if higher powers of $v_1(c)$ than the first are ignored, (24) becomes

$$\begin{aligned} P = \int_{b=0}^{1} T_{(f_1+f_2)}\left\{\sqrt{(f_1+f_2)}\sqrt{\left(\frac{\gamma b}{f_1}+\frac{(1-\gamma)(1-b)}{f_2}\right)}\,v_0(c)\right\} p(b)\,db \\ + \int_{b=0}^{1}\left[\sqrt{(f_1+f_2)}\sqrt{\left(\frac{\gamma b}{f_1}+\frac{(1-\gamma)(1-b)}{f_2}\right)}\,v_1(c)\right] \\ \times T'_{(f_1+f_2)}\left\{\sqrt{(f_1+f_2)}\sqrt{\left(\frac{\gamma b}{f_1}+\frac{(1-\gamma)(1-b)}{f_2}\right)}\,v_0(c)\right\} p(b)\,db. \end{aligned} \tag{27}$$

If we can obtain $v_1(c)$ to satisfy equation (27), then we might expect that $\{v_0(c)+v_1(c)\}$ will be an improved approximation to the solution of (24)—how much improved depending on the magnitude of the neglected terms v^2, etc. Now although (27) is a linear integral equation for $v_1(c)$ it is still not easy to solve, and to make rapid progress further simplification appears to be necessary. Such simplification may be introduced by applying to the second integral in (27) a certain crude approximative procedure which will now be explained.

The distribution, $p(b)\,db$, given in equation (12) has mean value at $b = f_1/(f_1+f_2)$ and has variance of order $1/f$ (here f is used collectively to denote f_1 and f_2). It follows that with large f the distribution will be closely concentrated about $f_1/(f_1+f_2)$ and that therefore an integral of the form $\int_0^1 g(b)\,p(b)db$ will be approximately equal to $g\left(\frac{f_1}{f_1+f_2}\right)$. This will be true at all events provided $g(b)$ is not changing in an awkward way in the vicinity of $f_1/(f_1+f_2)$.

Now when $b = f_1/(f_1+f_2)$ we have from (25), $c = \gamma$, and the above method applied to the second integral of (27) gives simply $v_1(\gamma)\,T'_{(f_1+f_2)}\{v_0(\gamma)\}$. If then I_0 is written for the first integral in (27) this equation yields

$$P = I_0 + v_1(\gamma)\,T'_{(f_1+f_2)}\{v_0(\gamma)\}, \tag{28}$$

i.e.

$$v_1(\gamma) = \frac{(P-I_0)}{T'_{(f_1+f_2)}\{v_0(\gamma)\}}. \tag{29}$$

For any given γ, I_0 can be obtained by quadrature and hence $(P-I_0)$. The quantity $T'_{(f_1+f_2)}\{v_0(\gamma)\}$ is simply the ordinate to the Student distribution with (f_1+f_2) degrees of freedom evaluated at the point $t = v_0(\gamma)$. We then obtain $v_1(\gamma)$ from (29). By taking a sufficient number of values of γ we can therefore tabulate $v_1(\gamma)$. Since $v_1(c)$ is the same function of c as $v_1(\gamma)$ is of γ this is equivalent to tabulating $v_1(c)$.

The function $v_1(c)$ thus obtained will not exactly satisfy (27) because of the crude approximation made to the second integral. Nevertheless, used in (26) this $v_1(c)$ may yet bring us appreciably closer to a solution of (24). With this fresh approximation we can start off a further cycle of the whole process and continue thus *ad lib.* The usefulness of the method must be judged by the way the successive approximations behave. Therefore before proceeding to further general discussion it may be advisable at this point to show how it works out in a particular case.

4. A PARTICULAR NUMERICAL TRIAL

Let us suppose that the sample sizes are $n_1 = 21$ and $n_2 = 21$, so that the estimates s_1^2 and s_2^2 of σ_1^2 and σ_2^2 are each based on 20 degrees of freedom. Suppose that we wish to calculate two-sided 90 % confidence limits for $(\bar{X}_1 - \bar{X}_2)$. This will be straightforward if we can find $v(c)$ such that $\Pr\{v < v(c)\} = 0{\cdot}95$. (For $P = 0{\cdot}05$ we have simply to change the sign of $v(c)$.)

From (24) the equation to be satisfied is

$$0{\cdot}95 = \int_{b=0}^{1} T_{40}\{\sqrt{[2\gamma b + 2(1-\gamma)(1-b)]}\, v(c)\}\, p(b)\, db, \tag{30}$$

where

$$p(b)\, db = \frac{1}{B(10, 10)}\, b^9 (1-b)^9\, db, \tag{31}$$

and it is understood in (30) that

$$c = \gamma b / \{\gamma b + (1-\gamma)(1-b)\}. \tag{32}$$

Since in large samples v is approximately normally distributed a first approximation to $v(c)$ is $v_0(c) = \xi$, where ξ is such that $N(\xi) = P$. (For $P = 0{\cdot}95$ we have $\xi = 1{\cdot}64485$.) We shall use this value to start our first cycle. Since this first approximation does not depend on c there is some simplification in the work which will not appear in later cycles.

We need to calculate first

$$I_0 = \Pr\{v < v_0(c)\} = \int_{b=0}^{1} T_{40}\{\sqrt{[2\gamma b + 2(1-\gamma)(1-b)]}\, (1{\cdot}64485)\}\, p(b)\, db. \tag{33}$$

for a sufficient number of values of γ. To begin with we have taken $\gamma = 0$, 0·1, 0·2, 0·3, 0·4 and 0·5. (Values for $\gamma = 0{\cdot}6$, etc., will follow by symmetry.)

Since I_0 is to be calculated by quadrature we must also choose a net of values of b. Again somewhat arbitrarily we have taken b by intervals of 0·1. With this net it is found in the present example that I_0 can be obtained to five decimal places for all γ. It is found, moreover, that the simple trapezoidal rule suffices to give this accuracy. (To satisfy ourselves on this point we have in certain instances doubled the number of ordinates and verified that the same answers are obtained.) The reason for the rather surprising accuracy of the trapezoidal rule lies in the fact that the ordinates contain the factor $p(b)$ which has high order contact at the ends of the range. The same method of quadrature works also satisfactorily on later cycles of the method.

The values of T_{40} corresponding to these γ and b are set out in Table 1. They were obtained by second-order interpolation into Pearson & Hartley's (1954) Table 9. In the final column of Table 1 is given $\frac{1}{10}p(b)$, and in the final line of the table is given I_0 obtained by quadrature from (33). (As has already been mentioned when $\gamma = 0$, v is actually distributed in the Student distribution with $f = 20$ and hence $\Pr(v < 1{\cdot}64485)$ is then equal to $T_{20}(1{\cdot}64485)$. Directly from tables this is seen to be 0·94219, confirming our quadratured value.)

Inspecting the values I_0 in Table 1 we see how $\Pr(v < 1{\cdot}64485)$ depends on γ (and hence on the ratio σ_1/σ_2). The value is never much different from $P = 0{\cdot}95$, which shows that the assumption that v is normally distributed about zero with unit standard deviation never has very serious consequences, whatever γ may be. Suppose, however, that we wish to choose $v(c)$ to bring $\Pr\{v < v(c)\}$ even nearer to 0·95. Then, following the method of the previous section, we need to calculate first

$$T'_{40}(v_0(\gamma)\} = \frac{0{\cdot}396457}{\left\{1 + \frac{[v_0(\gamma)]^2}{40}\right\}^{\frac{41}{2}}}. \tag{34}$$

Since $v_0(\gamma) = 1{\cdot}64485$ this gives 0·103634, whatever γ may be. Hence (29) yields

$$v_1(\gamma) = 9{\cdot}64931(P - I_0). \tag{35}$$

These values are given in Table 2. It will be noted that although $v_1(\gamma)$ is given to five decimal places it can only be correct to *four* since the $(P - I_0)$ are multiplied by about ten. The limitation on accuracy is imposed by the fact that the tables of T which we have used are given to five decimals and hence I_0 could only be calculated to five decimals.

Table 1. $T_{40}\{\sqrt{[2\gamma b + 2(1-\gamma)(1-b)]}\,(1{\cdot}64485)\}$

b \ γ	0	0·1	0·2	0·3	0·4	0·5	$\frac{1}{10}p(b)$
0·0	0·98743	0·98344	0·97802	0·97066	0·96045	0·94608	0·000000
0·1	·98344	·97926	·97390	·96698	·95796	·94608	·000036
0·2	·97802	·97390	·96888	·96277	·95530	·94608	·006348
0·3	·97066	·96698	·96277	·95796	·95243	·94608	·073374
0·4	·96045	·95796	·95530	·95243	·94937	·94608	·244045
0·5	·94608	·94608	·94608	·94608	·94608	·94608	·352394
0·6	0·92547	0·93024	0·93465	0·93874	0·94255	0·94608	0·244045
0·7	·89501	·90871	·92032	·93024	·93874	·94608	·073374
0·8	·84825	·87871	·90215	·92032	·93465	·94608	·006348
0·9	·76687	·83519	·87871	·90871	·93024	·94608	·000036
1·0	·50000	·76687	·84825	·89501	·92547	·94608	·000000
I_0	0·94219	0·94365	0·94474	0·94549	0·94594	0·94608	—

Table 2. *Calculation of first corrections*

γ	0	0·1	0·2	0·3	0·4	0·5
I_0	0·94219	0·94365	0·94474	0·94549	0·94594	0·94608
$P - I_0$	·00781	·00635	·00526	·00451	·00406	·00392
$v_1(\gamma)$	·07536	·06126	·05076	·04352	·03922	·03782

$P = 0{\cdot}95, \qquad v_1(\gamma) = 9{\cdot}64931(P - I_0).$

Table 3

c	0·0 or 1·0	0·1 or 0·9	0·2 or 0·8	0·3 or 0·7	0·4 or 0·6	0·5
$v_1(c)$	0·07536	0·06126	0·05076	0·04352	0·03922	0·03782
$v_0(c) + v_1(c)$	1·7202	1·7061	1·6956	1·6884	1·6841	1·6827

($(v_0(c) + v_1(c))$ equals new $v_0(c)$ for second cycle.)

In Table 3 the values of v_1 are rewritten against c and then added on to $v_0(c)$—in this case 1·64485 whatever c. The resulting sum is our next approximation to $v(c)$ and will form the new $v_0(c)$ for the second application of the method. At this state we have cut down to *four* decimal places, as this is all that can be guaranteed.

5. NUMERICAL TRIAL—SECOND CYCLE

Having obtained our new approximation, $v_0(c)$, to $v(c)$ the next step is to see how much improvement it has produced. This necessitates a recalculation of I_0 from the equation

$$I_0 = \Pr\{v < v_0(c)\} = \int_{b=0}^{1} T_{40}\{\sqrt{[2\gamma b + 2(1-\gamma)(1-b)]}\, v_0(c)\}\, p(b)\, db. \tag{36}$$

In this cycle $v_0(c)$ is no longer constant but changes with c where, for substitution in (36), c is given in terms of b and γ by (32). In Table 4, therefore, we exhibit the values of c corresponding to the sets of γ and b which we are using.

Table 4. $c = \gamma b\{\gamma b + (1-\gamma)(1-b)\}^{-1}$

b \ γ	0	0·1	0·2	0·3	0·4	0·5	
0·0	0	0·00000	0·00000	0·00000	0·00000	0·0	
0·1	0	·01220	·02703	·04545	·06897	0·1	
0·2	0	·02703	·05882	·09677	·14286	0·2	
0·3	0	·04545	·09677	·15517	·22222	0·3	
0·4	0	·06897	·14286	·22222	·30769	0·4	
0·5	0	·10000	·20000	·30000	0·40000	0·5	
0·6	0	0·14286	0·27273	0·39130	0·50000	0·6	
0·7	0	·20588	·36842	·50000	·60870	0·7	
0·8	0	·30769	·50000	·63158	·72727	0·8	
0·9	0	·50000	·69231	·79412	·85714	0·9	
1·0	—	1·00000	1·00000	1·00000	1·00000	1·0	

Table 5. $v_0(c)$ *for second cycle*

b \ γ	0	0·1	0·2	0·3	0·4	0·5
0·0	1·7202	1·7202	1·7202	1·7202	1·7202	1·7202
0·1	1·7202	1·7183	1·7160	1·7133	1·7101	1·7061
0·2	1·7202	1·7160	1·7115	1·7065	1·7012	1·6956
0·3	1·7202	1·7133	1·7065	1·6999	1·6937	1·6884
0·4	1·7202	1·7101	1·7012	1·6937	1·6879	1·6841
0·5	1·7202	1·7061	1·6956	1·6884	1·6841	1·6827
0·6	1·7202	1·7012	1·6900	1·6843	1·6827	1·6841
0·7	1·7202	1·6951	1·6851	1·6827	1·6843	1·6884
0·8	1·7202	1·6879	1·6827	1·6851	1·6900	1·6956
0·9	1·7202	1·6827	1·6879	1·6951	1·7012	1·7061
1·0	—	1·7202	1·7202	1·7202	1·7202	1·7202

If $\gamma = 0$ and $b = 1$ then c is indeterminate, but lack of knowledge of this does not affect our subsequent quadrature, since $p(b) = 0$ at $b = 1$.

The values of $v_0(c)$ corresponding to the c of Table 4 are obtained by interpolating into the last line of Table 3. The results are set out in Table 5, and the corresponding values of $T_{40}\{\sqrt{[2\gamma b + 2(1-\gamma)(1-b)]}\, v_0(c)\}$ are given in Table 6. (Values for $b = 0$ and $b = 1$ are not included as they contribute nothing to the quadratures.) The last line of Table 6 gives the values of I_0 calculated from (36).

It is clear from Table 6 that a great improvement has been made in I_0. Whereas previously we had $\Pr\{v < 1{\cdot}64485\}$ differing from 0·950 up to as much as 8 units in the third decimal place, we have now with our first correction $\Pr\{v < v_0(c)\} = 0{\cdot}950$ to three decimal places throughout.

Table 6. $T_{40}\{\sqrt{[2\gamma b + 2(1-\gamma)(1-b)]}\, v_0(c)\}$ *for second cycle*

b \ γ	0	0·1	0·2	0·3	0·4	0·5	$\frac{1}{10}p(b)$
0·1	0·98688	0·98320	0·97838	0·97201	0·96353	0·95213	0·000036
0·2	·98222	·97838	·97361	·96769	·96033	·95113	·006348
0·3	·97577	·97201	·96769	·96271	·95699	·95044	·073374
0·4	·96660	·96353	·96033	·95699	·95354	·95002	·244045
0·5	·95344	·95213	·95113	·95044	·95002	·94988	·352394
0·6	0·93411	0·93658	0·93963	0·94298	0·94647	0·95002	0·244045
0·7	·90488	·91507	·92518	·93455	·94298	·95044	·073374
0·8	·85845	·88467	·90708	·92518	·93963	·95113	·006348
0·9	·77688	·84062	·88467	·91507	·93658	·95213	·000036
I_0	0·94958	0·94959	0·94974	0·94990	0·95001	0·95005	—

Table 7. *Calculation of second corrections*

γ	0	0·1	0·2	0·3	0·4	0·5
I_0	0·94958	0·94959	0·94974	0·94990	0·95001	0·95005
$(P-I_0)$	·00042	·00041	·00026	0·00010	− ·00001	− ·00005
$[T'_{40}(v_0(\gamma))]^{-1}$	10·90	10·66	10·47	10·35	10·27	10·25
$v_1(\gamma)$	0·0045	0·0043	0·0027	0·0010	−0·0001	−0·0005

Table 8

c	0 or 1·0	0·1 or 0·9	0·2 or 0·8	0·3 or 0·7	0·4 or 0·6	0·5
$v_0(c)$	1·7202	1·7061	1·6956	1·6884	1·6841	1·6827
$v_1(c)$	0·0045	0·0043	0·0027	0·0010	−0·0001	−0·0005
$v_0(c)+v_1(c)$	1·7247	1·7105	1·6983	1·6894	1·6840	1·6822

$\{v_0(c)+v_1(c)\}$ equals new $v_0(c)$ for third cycle

Now applying our general procedure to improve this still further we have from (29) and (34)

$$v_1(\gamma) = (P-I_0)\,[T'_{40}\{v_0(\gamma)\}]^{-1} \tag{37}$$

$$= 2{\cdot}52234\left[1+\frac{\{v_0(\gamma)\}^2}{40}\right]^{41}(P-I_0).$$

The calculation of these corrections is shown in Table 7.

In the second line of Table 8 the v_1 are set out again against c and in the final line they are added on to $v_0(c)$ to give now our third approximation to $v(c)$.

6. NUMERICAL TRIAL—THIRD CYCLE AND DISCUSSION

A third cycle of calculations has been carried out starting with the new approximation derived in the previous section. This cycle will not, however, be described here in detail, since no new features are involved in the calculations. Instead, the final results of the third cycle are simply summarized in Table 9. This table also repeats the salient points from the earlier cycles so that a general view of the successive iterations may be obtained.

Table 9. *Summary of calculations for* $f_1 = f_2 = 20$

c	0 or 1·0	0·1 or 0·9	0·2 or 0·8	0·3 or 0·7	0·4 or 0·6	0·5
Initial approx. to $v(c)$	1·6449	1·6449	1·6449	1·6449	1·6449	1·6449
1st correction to $v(c)$	0·0754	0·0613	0·0508	0·0435	0·0392	0·0378
2nd approx. to $v(c)$	1·7202	1·7061	1·6956	1·6884	1·6841	1·6827
2nd correction to $v(c)$	0·0045	0·0043	0·0027	0·0010	−0·0001	−0·0005
3rd approx. to $v(c)$	1·7247	1·7105	1·6983	1·6894	1·6840	1·6822
3rd correction to $v(c)$	0·0000	0·0002	0·0002	0·0000	−0·0003	−0·0004
4th approx. to $v(c)$	1·7247	1·7107	1·6985	1·6894	1·6837	1·6818
γ	0 or 1·0	0·1 or 0·9	0·2 or 0·8	0·3 or 0·7	0·4 or 0·6	0·5
I_0 [i.e Pr $(v<v(c))$] for 1st approx.	0·94219	0·94365	0·94474	0·94549	0·94594	0·94608
$(0·95000 - I_0)$	·00781	·00635	·00526	·00451	·00406	·00392
I_0 [i.e. Pr $(v<v(c))$] for 2nd approx.	0·94958	0·94959	0·94974	0·94990	0·95001	0·95005
$(0·95000 - I_0)$	·00042	·00041	·00026	·00010	− ·00001	− ·00005
I_0 [i.e Pr $(v<v(c))$] for 3rd approx.	0·95000	0·94998	0·94998	0·95000	0·95003	0·95004
$(0·95000 - I_0)$	·00000	·00002	·00002	·00000	− ·00003	− ·00004
I_0 [i.e Pr $(v<v(c))$] for 4th approx.	All within $0·95000 \pm 0·00002$					

The top half of Table 9 shows the successive approximations to $v(c)$, whereas the bottom half shows the calculated values of $I_0 = \Pr\{v < v(c)\}$ which are obtained using these approximations. I_0 in each case depends on $\gamma = \sigma_1^2/(\sigma_1^2 + \sigma_2^2)$, and the object of the method is finally to produce a $v(c)$ such that $\Pr\{v < v(c)\} = 0·95000$, irrespective of γ. We see that after the third cycle all the I_0 have been brought to within 0·00002 of the required value.

We could attempt to improve upon this by proceeding to a further cycle, but the labour seems scarcely worth while since we are approaching a stage where rounding off errors may begin to count. If further accuracy were required it would strictly be necessary to work to an extra decimal place throughout and then round down at the end. As far as we have gone, the final approximation to $v(c)$, given to four decimals in Table 9, may be in error by two or three units in the last figure.

We shall return again later to the question of the accuracy which this numerical method

can provide. Before doing so, however, we shall show briefly in the next section how the method connects up with the original series solution upon which such tables as are so far published are based.

7. Development of solution in powers of $1/f$

The integral equation to be satisfied by $v(c)$ was given in a form involving a single integration in equation (24) above. The iterative method described in §3 was applied to the equation in this form. The same iterative method, however, could have been applied equally well in theory to the equation in the original form (22), involving a double integration over M_1 and M_2, viz.

$$P = \int_0^\infty \int_0^\infty N\{\sqrt{(M_1\gamma + M_2\overline{1-\gamma})}\, v(c)\}\, p(M_1)\, p(M_2)\, dM_1\, dM_2, \tag{38}$$

where in the expression $v(c)$ it is understood that

$$c = M_1\gamma\{M_1\gamma + M_2(1-\gamma)\}^{-1}. \tag{39}$$

Writing, as before, $v(c) = v_0(c) + v_1(c)$ in (38) and neglecting powers of v_1 higher than the first we obtain an equation similar to (27), viz.

$$\begin{aligned} P = &\int_0^\infty \int_0^\infty N\{\sqrt{(M_1\gamma + M_2\overline{1-\gamma})}\, v_0(c)\}\, p(M_1)\, p(M_2)\, dM_1\, dM_2 \\ &+ \int_0^\infty \int_0^\infty [\sqrt{(M_1\gamma + M_2\overline{1-\gamma})}\, v_1(c)] \times N'\{\sqrt{(M_1\gamma + M_2\overline{1-\gamma})}\, v_0(c)\}\, p(M_1)\, p(M_2)\, dM_1\, dM_2, \end{aligned} \tag{40}$$

i.e. $P = I_0 + I_1$ (say), if we use summary letters to denote the two integrals in (40).

Now the distribution, $p(M_1)\, dM_1$, has mean at $M_1 = 1$ and variance $2/f_1$. Similarly, M_2 has mean unity and variance $2/f_2$. With large f_1 and f_2, therefore, $p(M_1)\, p(M_2)\, dM_1\, dM_2$ is concentrated closely about the point $(1, 1)$. An integral of the form

$$\iint g(M_1, M_2)\, p(M_1)\, p(M_2)\, dM_1\, dM_2$$

will in these circumstances be approximately equal to $g(1, 1)$ provided $g(M_1, M_2)$ is not changing awkwardly near $(1, 1)$. If this approximation be applied to I_1 we have, instead of (40),

$$P = I_0 + v_1(\gamma)\, N'\{v_0(\gamma)\}, \tag{41}$$

i.e.

$$v_1(\gamma) = \frac{(P - I_0)}{N'\{v_0(\gamma)\}}. \tag{42}$$

This equation is similar to (29) and may be used in the same way as (29) to provide repeated corrections to the estimated value of $v(c)$. Instead of dividing $(P - I_0)$ by the ordinate to a Student distribution we are in (42) dividing it by the ordinate to a normal distribution. This, however, is a small detail. The main difference is that in (42), I_0 is a double integral over M_1 and M_2. This would present a formidable problem if it had to be calculated by quadrature. For numerical work it is undoubtedly better to transform to b and w by equations (9). However, if we wish to proceed differently and produce explicit algebraic expressions for the successive corrections to $v(c)$ in groups of terms involving successive powers of $1/f$, it is perhaps slightly easier to work from (42) where I_0 is conceived as an integral over M_1 and M_2.

Suppose $v_0(c)$ is an approximation to $v(c)$ which is correct to order f^{-r} (say), so that $v_1(c)$ is of order $f^{-(r+1)}$. (In general, expansions in $1/f$ of the nature we are discussing involve the half-integral powers, but owing to the symmetry of the present problem only the integral powers are involved.) In replacing (38) by (40) we are ignoring v_1^2, i.e. terms of order $f^{-2(r+1)}$. If we are content, then, to find $v_1(c)$ correct to the next order $f^{-(r+1)}$ we shall be committing no error in using (40). Now the second integral, I_1, in (40), which involves $v_1(c)$ linearly, is clearly of order $f^{-(r+1)}$ and hence the replacement of it by the approximation used in (41) introduces an error of higher order than $f^{-(r+1)}$. To order $f^{-(r+1)}$ equation (41) is still equivalent to the original integral equation (38). Hence (42) will give a correction to $v(c)$ of order $f^{-(r+1)}$, the error at this stage now becoming of order $f^{-(r+2)}$. The iterative process, then, corresponds to finding corrective terms of successive orders in $1/f$.

To make the transition finally to an algebraic form we proceed to express I_0 in (42) itself as an expansion in powers of $1/f$. The integral of any expression $g(M_1, M_2)$ over the distribution of M_1 and M_2 may be expressed formally as follows

$$G = \iint g(M_1, M_2)\, p(M_1)\, p(M_2)\, dM_1\, dM_2$$

$$= \iint e^{(M_1-1)\partial_1+(M_2-1)\partial_2}\, g(w_1, w_2)\, p(M_1)\, p(M_2)\, dM_1\, dM_2, \tag{43}$$

where it is understood that the exponential is to be expanded in a power series in ∂ and that

$$\partial_1^s \partial_2^t g(w_1, w_2) = \left[\frac{\partial^{s+t}}{\partial w_1\, \partial w_2}\, g(w_1, w_2)\right]_{w=1}. \tag{44}$$

We are simply expanding $g(M_1, M_2)$ formally in a Taylor Series about the point $(1, 1)$.

Now (43) can be integrated out with respect to M_1 and M_2 to yield finally

$$G = \Theta g(w_1, w_2), \tag{45}$$

where

$$\Theta = \prod_{i=1}^{2} \left[\left(1-\frac{2\partial_i}{f_i}\right)^{-\frac{1}{2}f} \exp(-\partial_i)\right]$$

$$= \exp\left[\frac{\partial_i^2}{f_i} + \tfrac{4}{3}\Sigma\frac{\partial_i^3}{f_i^2} + 2\Sigma\frac{\partial_i^4}{f_i^3} + \text{etc.}\right]$$

$$= 1 + \Sigma\frac{\partial_i^2}{f_i} + \left\{\tfrac{4}{3}\Sigma\frac{\partial_i^3}{f_i^2} + \frac{1}{2}\left(\Sigma\frac{\partial_i^2}{f_i}\right)^2\right\} + \text{etc.} \tag{46}$$

If we write $\Theta_{(r+1)}$ for the expansion (46) carried as far as terms of order $f^{-(r+1)}$, then $\Theta_{(r+1)} g(w_1, w_2)$ will be equal to G to order $f^{-(r+1)}$ (provided that the differentiations (44) do not produce of their own accord any powers of f greater than zero). In particular therefore we have to order $f^{-(r+1)}$

$$I_0 = \Theta_{(r+1)} N\left\{\sqrt{[w_1\gamma + w_2(1-\gamma)]}\, v_0\left(\frac{w_1\gamma}{w_1\gamma + w_2(1-\gamma)}\right)\right\}. \tag{47}$$

If we substitute this in equation (42) we shall therefore have an expression for the correction v_1 correct to order $f^{-(r+1)}$. To the order concerned this can be simplified by making use of the fact that v_0 differs from ξ by something of the order $1/f$. Hence to order $f^{-(r+1)}$ we can write (42) as

$$N'(\xi)\, v_1(\gamma) = N(\xi) - \Theta_{(r+1)} N\left\{\sqrt{[w_1\gamma + w_2(1-\gamma)]}\, v_0\left(\frac{w_1\gamma}{w_1\gamma + w_2(1-\gamma)}\right)\right\}. \tag{48}$$

Successive applications of this equation lead to the series solution. Thus, for example, if $r = 0$ we can start with the large sample approximation $v_0(c) = \xi$ and (48) gives

$$N'(\xi)\, v_1(\gamma) = N(\xi) - \left\{1 + \left(\frac{\partial_1^2}{f_1} + \frac{\partial_2^2}{f_2}\right)\right\} N\{\xi\sqrt{[w_1\gamma + w_2(1-\gamma)]}\}. \tag{49}$$

Carrying out the differential operations this leads to

$$v_1(\gamma) = \tfrac{1}{4}\xi(1+\xi^2)\left\{\frac{\gamma^2}{f_1} + \frac{(1-\gamma)^2}{f_2}\right\}. \tag{50}$$

Our first correction to the large sample approximation is therefore

$$v_1(c) = \tfrac{1}{4}\xi(1+\xi^2)\left\{\frac{c^2}{f_1} + \frac{(1-c)^2}{f_2}\right\}, \tag{51}$$

and the resulting second approximation is

$$v(c) = \xi + \tfrac{1}{4}\xi(1+\xi^2)\left\{\frac{c^2}{f_1} + \frac{(1-c)^2}{f_2}\right\}. \tag{52}$$

This second approximation may now be taken as our new $v_0(c)$ and be substituted in (48), putting $r = 1$. This produces (Welch, 1947) the next corrective term

$$v_1(c) = -\tfrac{1}{2}\xi(1+\xi^2)\left\{\frac{c^2}{f_1} + \frac{(1-c)^2}{f_2}\right\} + \tfrac{1}{3}(3\xi + 5\xi^3 + \xi^5)\left\{\frac{c^3}{f_1} + \frac{(1-c)^3}{f_2}\right\}$$
$$-\tfrac{1}{32}(15\xi + 32\xi^3 + 9\xi^5)\left\{\frac{c^2}{f_1} + \frac{(1-c)^2}{f_2}\right\}^2. \tag{53}$$

Two further corrective terms are given by Alice A. Aspin (1948).

8. Series solution when $f_1 = f_2 = 20$

The explicit expressions for the corrective terms given in the preceding section will not, of course, have the same numerical values as those of the corrective terms given by the method described earlier in the paper. But sufficient has been said about orders of magnitude to indicate that they should exhibit the same general sort of behaviour. It will be of interest, therefore, to see what the series leads to in the special case $f_1 = f_2 = 20$ which has been discussed at length earlier. Taking $P = 0{\cdot}95$ (i.e. $\xi = 1{\cdot}64485$), we find for the successive series corrections the values shown in Table 10.

Table 10 may be compared directly with the upper half of Table 9. We do not now possess the values of $\Pr\{v < v(c)\}$ at each stage as we did in the lower half of Table 9. The general parallelism between such figures as are given, however, suffices to show that the series method is doing essentially the same thing as the other method, namely, at each stage bringing $\Pr\{v < v(c)\}$ closer to 0·95. The final approximation given in Table 10 is unlikely to be in error more than one unit in the last figure shown, and the corresponding

$$\Pr\{v < v(c)\}$$

is unlikely to differ by more than 0·00001 from 0·95000, whatever the true γ may chance to be.

Table 10. *Series calculations for* $f_1 = f_2 = 20$ $(P = 0{\cdot}95)$

c	0 or 1·0	0·1 or 0·9	0·2 or 0·8	0·3 or 0·7	0·4 or 0·6	0·5
Initial approx. to $v(c)$	1·6449	1·6449	1·6449	1·6449	1·6449	1·6449
1st correction to $v(c)$	0·0762	0·0625	0·0518	0·0442	0·0396	0·0381
2nd approx. to $v(c)$	1·7211	1·7073	1·6967	1·6890	1·6845	1·6829
2nd correction to $v(c)$	0·0035	0·0031	0·0019	0·0004	−0·0006	−0·0010
3rd approx. to $v(c)$	1·7246	1·7105	1·6985	1·6895	1·6838	1·6819
3rd correction to $v(c)$	0·0001	0·0002	0·0002	−0·0000	−0·0003	−0·0004
4th approx. to $v(c)$	1·7247	1·7107	1·6987	1·6895	1·6836	1·6816
4th correction to $v(c)$	0·0000	0·0000	0·0000	0·0000	0·0000	−0·0001
5th approx. to $v(c)$	1·7247	1·7107	1·6987	1·6895	1·6835	1·6815

9. FURTHER DISCUSSION

The foregoing comparisons have underlined the iterative nature of the series solution. In the case which we have exemplified, viz. $f_1 = f_2 = 20$, the final $v(c)$, which includes terms of order f^{-4}, was seen to give $\Pr\{v < v(c)\} = 0{\cdot}95000$ with perhaps an error of 1 unit in the fifth place. The successive inclusion of terms of orders $f^0, f^{-1}, f^{-2}, f^{-3}$ and f^{-4} represents successive stages in the progress of $\Pr\{v < v(c)\}$ towards this goal. It cannot be assumed, however, that such iteration towards a result of specified accuracy will take place whatever f_1 and f_2 may be. One's expectation that the method will iterate rapidly (or, perhaps, will even iterate at all) to the required result naturally diminishes as f_1 and f_2 diminish.

Some consideration of the point at which the series ceases to be useful has been given by Aspin (1948) and by Welch (cf. Aspin, 1949, Appendix). The conclusions to be drawn from their work differ somewhat according to where the emphasis is placed. They depend on whether the problem is regarded as that of obtaining $v(c)$ itself to a certain prescribed accuracy or whether it is regarded as that of satisfying $\Pr\{v < v(c)\} = P$ to a certain degree of accuracy. In the tabulation so far carried out (Aspin, 1949), the general aim was to tabulate $v(c)$ correct to two decimal places throughout. It appeared then that to attain this result with the series for the case $P = 0{\cdot}95$, for instance, it would be inadvisable to let either f_1 or f_2 fall below 6. This limitation was not imposed by the fact that the series had only been worked out to terms of order f^{-4}. It could be seen from the behaviour of the terms already known that, however far the series were carried it was unlikely to provide in general stable values of $v(c)$ to more than two decimal places when either f_1 or f_2 was less than 6.

Turning now to the other way of regarding the problem, direct calculation shows that the values of $\Pr\{v < v(c)\}$ obtained at $f_1 = f_2 = 6$ by using the two-decimal $v(c)$, never differ from 0·9500 by more than two units in the fourth place, whatever γ. It is conceivable that for *practical* purposes one would be content with a solution solving our basic equation less accurately than this. It might be considered sufficient, for instance, to make

$$\Pr\{v < v(c)\} = 0{\cdot}950$$

to three decimal places only. If this were the case then undoubtedly a solution could be obtained from the series for some values of f_1 and f_2 lower than 6. The series cut off at some

point (say after the f^{-3} or f^{-4} terms) would provide a $v(c)$ which could, if we wished, be written down to two decimal places and which would have the property that

$$\Pr\{v < v(c)\} = 0{\cdot}950$$

to three decimals. This $v(c)$, however, although written down to two places, could not be said to be *correct* to two places in the sense that it was unique. For carrying the series to more terms might give another $v(c)$ differing in the second decimal but yet still satisfying $\Pr\{v < v(c)\} = 0{\cdot}950$ to three decimals. It is debatable whether in such circumstances it would be worth while attempting in the table to give $v(c)$ to more than one place.

From the practical point of view some of the considerations outlined above may seem somewhat unreal. With very small f_1 and f_2 one is perhaps not likely to worry much in any practical circumstances about anything after the first place in $v(c)$. Other considerations, however, affect the tabulator, who may feel that a table with so few figures looks rather thin. At all events it seems worth while to explore what results can be obtained with other iterative procedures. The numerical method which was described at the beginning of this paper is, as we have shown, very much akin to the series method and therefore, perhaps, one should not expect very much advance to be made by using it. There are, however, various modifications which can be made of the numerical method as we have described it. These modifications seem to offer the possibility of stronger iteration. Until they have been tried out in some detail it would seem to us rather premature to attempt at the moment to predict what further progress can be made on these lines in extending Mrs Aspin's two-decimal table of $v(c)$ to values of f_1 and f_2 lower than those already given.

10. Summary

The connexion has been shown between two methods of calculating critical values used in the problem of comparing two mean values. The methods are: (i) a predominantly numerical method and (ii) a method involving first the expression of the critical values in a series of groups of algebraic terms of different orders. This latter method has been already used in the computation of tables.

The close connexion between the two methods helps to emphasize what the series method is doing. A particular case (viz. $f_1 = f_2 = 20$) is discussed at some length, the important comparisons being between the figures given in Tables 9 and 10.

REFERENCES

Aspin, A. A. (1948). An examination and further development of a formula occurring in the problem of comparing two mean values. *Biometrika*, **35**, 88–96.

Aspin, A. A. (1949). Tables for use in comparisons whose accuracy involves two variances, separately estimated (with an Appendix by B. L. Welch). *Biometrika*, **36**, 290–96. Reprinted as Table 11 of Pearson & Hartley (1954).

Pearson, E. S. & Hartley, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge: The University Press for the Biometrika Trustees.

Welch, B. L. (1947). The generalization of 'Student's' problem when several different population variances are involved. *Biometrika*, **34**, 28–35.

SOME FURTHER RESULTS IN THE THEORY OF PEDESTRIANS AND ROAD TRAFFIC

By ALAN J. MAYNE

1. Introduction and summary

Tanner (1951) has considered several statistical problems about the passage of pedestrians across a road, on the assumption that vehicles and pedestrians arrive at random. Here his main results are generalized to the case where the time intervals between successive arrivals have a given distribution, any two such intervals being independently distributed. Tanner's results follow for the special case when the interval distributions are negative exponential, which occurs when there is a random flow of traffic. The size of groups of pedestrians on an island between two lanes of traffic is also considered. The effect of the presence of islands on pedestrian delay is discussed, and it is shown that, under certain conditions, the 'efficiency' in two specially defined senses can be multiplied by at least r^2, by the introduction of $(r-1)$ islands suitably spaced in the middle of the road. Finally, the results are given of a further examination of some empirical traffic data, which supplements the work of Adams (1936, 1948).

2. Determination of the distribution of pedestrian delay

Suppose that there is a single lane of traffic, and that a pedestrian is able to cross, at any given instant, if no vehicle is due to arrive less than time I after that instant. If $t_1, t_2, \ldots, t_n, \ldots$ are the arrival times of vehicles at the crossing-point, then it will be assumed that $(t_2-t_1), \ldots, (t_n-t_{n-1}) \ldots$ are statistically independent, and that they are all distributed as

$$x(u) \equiv dX(u) \quad (0<u<\infty;\ X(0)=0). \tag{1}$$

Let T be the delay to the pedestrian before crossing is possible. Let the probability that this delay lies between T and $(T+dT)$ be

$$f(T)\,dT \equiv dF(T) \quad (T>0), \tag{2}$$

and the probability of zero delay be $F(0)$.

In order to write the expression for $F(T)$ in a compact form, and in order to simplify its derivation, introduce the following notation:

$$\left.\begin{aligned}
&h(t;s) \equiv \int_0^I e^{-sv}x(v+t)\,dv \equiv \int_0^I e^{-sv}d_vX(v+t),\\
&k(t;s) \equiv \int_I^\infty e^{-sv}x(v+t)\,dv \equiv \int_I^\infty e^{-sv}d_vX(v+t),\\
&\left.\begin{aligned} h(s) &\equiv h(0;s), \quad h^{(i)}(s) \equiv \frac{d^i}{ds^i}h(s)\\ k(s) &\equiv k(0;s), \quad k^{(i)}(s) \equiv \frac{d^i}{ds^i}k(s)\end{aligned}\right\} \quad (i=1,2,\ldots),\\
&\left.\begin{aligned} h &\equiv h(0), \quad h^{(i)} \equiv h^{(i)}(0)\\ k &\equiv k(0), \quad k^{(i)} \equiv k^{(i)}(0)\end{aligned}\right.\\
&j(s) \equiv \frac{1}{1-h(s)},\\
&\text{Mean} \qquad m \equiv -[k^{(1)}+h^{(1)}] \equiv \int_0^\infty v\,dX(v).
\end{aligned}\right\} \tag{3}$$

THEOREM 1. (1) The probability of zero delay is

$$F(0) = -\frac{1}{m}[Ik + k^{(1)}]. \tag{4}$$

(2) The Laplace transform of the distribution of pedestrian delay is

$$L\{s, T; f(T)\} \equiv \int_0^\infty e^{-sT}\, dF(T) = F(0) + \frac{k}{ms}[1 - k e^{-Is} j(s)]$$
$$= \frac{1}{ms}\{k[1 - Is - k e^{-Is} j(s)] - k^{(1)} s\}. \tag{5}$$

(3) The mean delay is

$$E(T) = \frac{1}{2m}\{I^2 k - [2h^{(1)} I - h^{(2)}] + 2h^{(1)2} k^{-1}\}. \tag{6}$$

(4) The variance of the delay is

$$\text{var}\,(T) = \frac{1}{3m}\left\{\begin{matrix} I^3 k - [3h^{(1)} I^2 - 3h^{(2)} I + h^{(3)}] \\ + 6h^{(1)}[h^{(1)} I - h^{(2)}]\, k^{-1} - 6h^{(1)3} k^{-2} \end{matrix}\right\} - \{E(T)\}^2. \tag{7}$$

Proof of Theorem 1. In order to calculate $F(T)$, it is convenient to find, first, the distribution of the delay, when the interval between the time of arrival of the pedestrian and the instant at which the vehicle passes the crossing-point has a known value t. For any given t, let

$$f(T \mid t)\, dT \equiv d_T F(T \mid t) \tag{8}$$

be the probability that the delay before crossing is possible lies between T and $(T + dT)$; let $F(0, \mid t)$ be the probability that this delay is zero.

In practice, however, t is not known, and is a random variable. Let its distribution be

$$\phi(t) \equiv d\Phi(t). \tag{9}$$

Then

$$F(T) = \int_0^\infty F(T \mid t)\, d\Phi(t). \tag{10}$$

The proof of Theorem 1 proceeds by finding expressions for $\Phi(t)$ and $F(T \mid t)$, whose values are given in two lemmas, and combining these expressions in accordance with equation (10). The value so obtained is then rearranged, so as to yield parts (1) and (2) of the Theorem; parts (3) and (4) are derived directly from part (2).

LEMMA 1.
$$\phi(t)\, dt \equiv d\Phi(t) = \frac{1}{m}[1 - X(t)]\, dt. \tag{11}$$

Proof of Lemma 1. Consider the interval between the vehicle-passage times preceding and following the arrival of the pedestrian at the crossing-point. Let the probability that this interval lies in $(u, u + du)$ be

$$y(u)\, du \equiv dY(u). \tag{12}$$

To calculate $Y(u)$, consider an infinitely long series of successive vehicle-intervals. A proportion $dX(u)$ of the intervals of this series have length between u and $(u + du)$, so that a proportion $m^{-1} u\, dX(u)$ of the length of the series is composed of intervals of length between u and $(u + du)$. As the arrival time of the pedestrian may be considered to be random with respect to the series of intervals between vehicle-passages, the probability of arriving during an interval of length between u and $(u + du)$ is

$$dY(u) = (\text{const.}) \times [dX(u)] \times u = \frac{1}{m} u\, dX(u), \tag{13}$$

since

$$\int_0^\infty \text{const.}\, u\, dX(u) = 1.$$

Since the arbitrary point lies in an interval of length u, and is distributed at random in that interval, then the probability that it lies at distance $(t, t+dt)$ from any given end of that interval, given u, is

$$u^{-1}dt, \quad \text{if} \quad 0<t\leqslant u;$$
$$0, \quad \text{if} \quad t>u. \tag{14}$$

Then $$d\Phi(t) = m^{-1}dt\int_t^\infty dX(u) = m^{-1}[1-X(t)]\,dt,$$

which is the requred result.†

LEMMA 2. $$L\{s,T;f(T\,|\,t)\} \equiv \int_0^\infty e^{-sT}\,d_T F(T\,|\,t)$$
$$= F(0\,|\,t) + \frac{kh(t;s)j(s)}{1-X(t)}. \tag{15}$$

Proof of Lemma 2. Suppose that a pedestrian arrives at time t after the last vehicle-passage. Then, if u is the difference between the arrival time of the next vehicle to pass after the arrival of the pedestrian, and the arrival time of the pedestrian, its distribution is

$$\frac{d_u X(u+t)}{1-X(t)}.$$

The probability that crossing is immediately possible is equal to the probability that the next vehicle will arrive at a time more than I later than the pedestrian's arrival, i.e. more than $(t+I)$ after the passage of the last vehicle. The required probability is

$$F(0\,|\,t) = \frac{1-X(t+I)}{1-X(t)}. \tag{16}$$

Next, consider what happens if crossing is not immediately possible.

If T is between 0 and I, then the crossing may occur immediately after the next vehicle-passage; this event has probability

$$\frac{kd_T X(T+t)}{1-X(t)},$$

since $$F(0\,|\,0) = k.$$

If the next vehicle-passage takes time $T>I$, the pedestrian could have crossed at once, which leads to a contradiction, Hence $T<I$ for the crossing to be possible immediately after the next vehicle-passage.

If T is between 0 and I, the only other way for the crossing to occur is for there to be a further delay after the arrival of the next vehicle. At this moment, the situation is identical with the initial situation when t equals 0. This follows from the original assumptions of the problem. Thus this event has probability

$$\frac{dT\int_0^{T-} f(T-u\,|\,0)\,d_u X(u+t)}{1-X(t)}.$$

If T is greater than I, the probability is calculated analogously and is

$$\frac{dT\int_0^{I} f(T-u\,|\,0)\,d_u X(u+t)}{1-X(t)}.$$

† This proof is similar to the argument of Malmquist (1947).

In consequence, $f(T\,|\,t)$ satisfies the equations

$$\left.\begin{aligned}[1-X(t)]f(T\,|\,t) &= kx(T+t)+\int_0^{T-} f(T-u\,|\,0)\,d_uX(u+t), \quad \text{if} \quad 0<T<I,\\ [1-X(t)]f(T\,|\,t) &= \int_0^{T} f(T-u\,|\,0)\,d_uX(u+t), \qquad\qquad \text{if} \quad T\geqslant I.\end{aligned}\right\} \tag{17}$$

Let $$\epsilon(T)\equiv 1, \quad \text{if} \quad 0\leqslant T<I; \qquad 0 \quad \text{if} \quad T\geqslant I.$$

Then (17) can be written as

$$[1-X(t)]f(T\,|\,t) = kx(T+t)\,\epsilon(T)+\int_0^{T-} f(T-u\,|\,0)\,x(u+T)\,\epsilon(u)\,du,$$

if $T>0$. In consequence,

$$\begin{aligned}L^*\{s,T;f(T\,|\,t)\} &\equiv \int_{0+}^{\infty} e^{-sT}\,d_TF(T\,|\,t)\\ &= \frac{k\int_{0+}^{I} e^{-sT}x(T+t)\,dT+\int_{0+}^{\infty}e^{-sT}\,dT\int_0^{T-} f(T-u\,|\,0)\,\epsilon(u)\,d_uX(u+t)}{1-X(t)}\\ &= \frac{k\int_{0+}^{I} e^{-sT}x(T+t)\,dT+\int_{0+}^{\infty}\epsilon(u)\,d_uX(u+t)\int_{0+}^{\infty}e^{-s(u+v)}f(v\,|\,0)\,dv}{1-X(t)}\end{aligned}$$

putting $v=T-u$ and changing the order of integration in the double integral,

$$= \frac{k\int_{0+}^{I} e^{-sT}\,d_TX(T+t)+\int_{0+}^{\infty}e^{-su}\epsilon(u)\,d_uX(u+t)\int_{0+}^{\infty}e^{-sT}\,d_TF(T\,|\,0)}{1-X(t)}, \tag{18}$$

replacing v by T in the double integral. Thus

$$L^*\{s,T;f(T\,|\,0)\} = kh(s)\,j(s), \tag{19}$$

putting $t=0$ in (18) and using (3). Also

$$L^*\{s,T;f(T\,|\,t)\} = \frac{kh(t;s)\,j(s)}{1-X(t)}, \tag{20}$$

applying (19) to (18) and again using (3), so that

$$L\{s,T;f(T\,|\,t)\} = F(0\,|\,t)+L^*\{s,T;f(T\,|\,t)\},$$

which, by (20), yields Lemma 2.

Completion of the proof of Theorem 1. By (10),

$$\begin{aligned}L\{s,T;f(T)\} &= \int_0^{\infty} L\{s,T;f(T\,|\,t)\}\,d\Phi(t)\\ &= \int_0^{\infty} F(0\,|\,t)\,d\Phi(t)+kj(s)\int_0^{\infty}[1-X(t)]^{-1}h(t;s)\,d\Phi(t)\\ &= F(0)+m^{-1}kj(s)\int_0^{\infty}dt\int_0^{I}e^{-sv}\,dX_v(v+t),\end{aligned} \tag{21}$$

by applying Lemmas 1 and 2. Now

$$\begin{aligned}F(0) &= m^{-1}\int_0^{\infty}dt\int_{t+I}^{\infty}dX(v) = m^{-1}\int_I^{\infty}dX(v)\int_0^{v-I}dt\\ &= m^{-1}\int_I^{\infty}(v-I)\,dX(v) = m^{-1}[-k^{(1)}-Ik],\end{aligned}$$

by (3), from which part (1) of the theorem follows immediately.

Again, $\int_0^\infty dt \int_0^I e^{-sv}\, d_v X(v+t) = \int_0^\infty dt \int_t^{t+I} e^{-s(u-t)}\, dX(u)$

$$= \int_0^I dX(u) \int_0^u e^{-s(u-t)}\, dt + \int_I^\infty dX(u) \int_{u-I}^u e^{-s(u-t)}\, dt$$

$$= s^{-1} \int_0^I (1 - e^{-su})\, dX(u) + s^{-1} \int_I^\infty (1 - e^{-Is})\, dX(u)$$

$$= s^{-1}[1 - h(s) - k e^{-Is}] \quad \text{by (3).} \tag{22}$$

By applying (22) to (21), the first form of part (2) of the theorem follows immediately. The second form of it is obtained by applying part (1) to the first form.

Using the fact that

$$j(s) = \left[k - \sum_{i=1}^{\infty} (i!)^{-1} h^{(i)} s^i\right]^{-1}, \tag{23}$$

$L\{s, T; f(T)\}$ can be expanded in a series of ascending powers of s. The coefficient of $(-s)^r$ gives the rth moment of the distribution of T multiplied by $1/r!$. By calculating these coefficients for $r = 1$ and 2, parts (3) and (4), respectively, of the theorem can be derived directly. Other statistics of the distribution, based on the moments, can be determined similarly.

The Laplace transform of $f(T)$ can most conveniently be inverted by expressing it in partial fractions. It can then be shown that $f(T)$ can be expanded as a series of negative exponential terms in T, multiplied by factors of the form $(\alpha \cos \gamma T + \beta \sin \gamma T)$, where α, β, γ are constants. These latter factors may sometimes be equal to one.

This follows because

$$L(s, u; x(u)) = X(0) + \frac{a(s)}{b(s)},$$

where $a(s)$ is analytic, $b(s)$ is of form $\prod_{i=1}^{\infty} (s - c_i)^{-r_i}$, and $a(s)$, $b(s)$ have no roots in common.

If all the roots c_i of $b(s)$ are simple (i.e. $r_i = 1$ for all i), then

$$\frac{a(s)}{b(s)} = \sum_{i=1}^{\infty} \frac{a(c_i)}{b'(c_i)\,(s - c_i)}, \tag{24}$$

whose inverse Laplace transform is a series of terms of the required form.

The first few terms of this series, or even the first term by itself, can usefully be taken as an asymptotic expansion for large enough T. Also, the values of T, such that $F(T) = p$, can be investigated. Full details are given in Mayne (1951).

For the special case of random flow, considered by Tanner (1951),

$$x(u) = N e^{-Nu}, \quad X(u) = 1 - e^{-Nu}, \tag{25}$$

where N is the *vehicle flow*, the mean number of vehicle-arrivals per unit time. Write

$$q \equiv NI, \quad a \equiv q e^{-q}.$$

THEOREM 2. If the traffic flow is random,

(1) The Laplace transform of the distribution of delay is

$$L\{s, T; f(T)\} = \frac{(s + N) e^{-NI}}{[s + N e^{-(s+N)I}]}. \tag{26}$$

(2) The distribution of the delay is given by the following expressions:

$$F(0) = e^{-NI} \equiv e^{-q},$$
$$f(T) = N e^{-NI} \equiv aI^{-1} \equiv A, \quad \text{if} \quad 0 < T \leqslant I, \tag{27}$$
$$f(T) = A\left\{\sum_{i=0}^{r-1} (i!)^{-1}(-A)^i[(T-iI)^i - e^{-q}(T-\overline{i+1}I)^i] + (r!)^{-1}(-A)^r(T-rI)^r\right\},$$
$$\text{if} \quad rI < T \leqslant (r+1)I.$$

(3) The mean delay is

$$E(T) = \frac{e^{NI} - NI - 1}{N} = \frac{(e^q - q - 1)I}{q}. \tag{28}$$

(4) The variance of the delay is

$$\text{var}\,(T) = \frac{e^{2NI} - 2NI\,e^{NI} - 1}{N^2} = \frac{(e^{2q} - 2q\,e^q - 1)I^2}{q^2}. \tag{29}$$

Proof. Theorem 2 is derived from Theorem 1 by straightforward calculation.†

Carrying out the partial fractions expansion described above, it is found that, for large enough T, and using (24),

$$f(T) \sim \frac{(q-q')\,e^{-q}\,e^{-N'T}}{(1-q')\,I}, \tag{30}$$

where $N' = q'I^{-1}$, and q' is that real root of $x\,e^{-x} = q\,e^{-q}$, which is not equal to q, except when $q = 1 = q'$.

It follows from (29) that the median $m_e(T) = 0$, if $F(0) \geqslant \frac{1}{2}$, i.e.

$$q \leqslant \log 2 \doteq 0{\cdot}6931. \tag{31}$$

If $e^{-q} \leqslant \frac{1}{2}$, but $e^{-q}(1+q) \geqslant \frac{1}{2}$, i.e. $0{\cdot}6931 \leqslant q \leqslant 1{\cdot}6784$, then

$$m_e(T) = N^{-1}(\tfrac{1}{2}e^q - 1). \tag{32}$$

For greater q,

$$m_e(T) \doteq \frac{I}{q'}\left\{\log\left[\frac{2(q-q')}{q'(1-q')}\right] - q\right\}, \tag{33}$$

a result derived directly from the asymptotic formula (30).

3. Optimum positions of islands to obtain minimum mean delay

Consider a road with r lanes of traffic, in each of which the traffic flow is random. Let the flow be N_i in the ith lane. At first, suppose that there is no island and that the safe-crossing time is I. Let $N = \sum_{i=1}^{r} N_i$ $(N_i > 0)$. Then, since the traffic flow in all the lanes is random, the mean delay is

$$\overline{T} = \frac{e^{NI} - NI - 1}{N}. \tag{34}$$

Now suppose that $(r-1)$ islands are placed between the lanes, such that the respective safe-crossing times are $I_1, \ldots, I_r$, where $I = \sum_{i=1}^{r} I_i$. These times are supposed proportional to the relative road widths, so that siting can be effected after the optimum $I_1, \ldots, I_r$ are chosen. The new mean delay is

$$\overline{T}' = \sum_{i=1}^{r} \frac{e^{N_i I_i} - N_i I_i - 1}{N_i}. \tag{35}$$

† The results of Theorem 2 were derived by Tanner (1951), who proved them by a different method. As Tanner points out on p. 384 of his paper, Garwood (1939) obtained a formula similar to (27), while considering a problem closely related to that of pedestrian delay. Equation (28) was first obtained by Adams (1936).

LEMMA 3. The minimum of $\overline{T}'$, for all positive $I_1, \ldots, I_r$ such that $I = \sum_{i=1}^{r} I_i$ is

$$\overline{T}'^* = \frac{e^{N^*I} - N^*I - 1}{N^*}, \tag{36}$$

where $$N^* = \left[\sum_{i=1}^{r} N_i^{-1}\right]^{-1}.$$

Proof. Replacing I_r by $I - \sum_{i=1}^{r-1} I_i$, it follows that

$$\frac{\partial \overline{T}'}{\partial I_i} = e^{N_i I_i} - \exp\left\{N_r\left[I - \sum_{i=1}^{r-1} I_i\right]\right\} \quad (i = 1 \text{ to } (r-1)).$$

Equating these derivatives to zero, it is found that, for the turning point,

$$N_i I_i = N_r\left[I - \sum_{i=1}^{r-1} I_i\right] = N_r I_r,$$

so that, in fact, $N_i I_i = N^* I$. Inserting these values of I_i into $\overline{T}'$, the value of $\overline{T}'^*$ is obtained. This value is in fact the true minimum of $\overline{T}'$, since the values of

$$\frac{\partial^2 T}{\partial I_i \partial I_j} = N_r e^{N_r I_r} + \delta_{ij} N_j e^{N_j I_j}$$

are, for all real, finite, $I_1, \ldots, I_r$, the coefficients of a positive-definite quadratic form.

LEMMA 4. The minimum value of NN^{*-1}, for all positive $N_1, \ldots, N_r$ such that $N = \sum_{i=1}^{r} N_i$, is r^2.

Proof. By Schwartz's inequality,

$$NN^{*-1} = \sum_{i=1}^{r} (N_i^{\frac{1}{2}})^2 \sum_{i=1}^{r} (N_i^{-\frac{1}{2}})^2 \geqslant \left[\sum_{i=1}^{r} (N_i^{\frac{1}{2}})(N_i^{-\frac{1}{2}})\right]^2 = r^2,$$

which is the required result, since equality holds for $N_i = Nr^{-1}$ ($i = 1$ to r).

Applying Lemmas 3 and 4, it is clear that the mean delay, for total flow N, with $(r-1)$ islands present, is at most equal to the mean delay, for total flow Nr^{-2}, with the islands absent.

Thus, for a given mean delay, the flow, with islands present, and suitably spaced, is at least r^2 times the flow, without islands.

Since $$\overline{T}(N) = \frac{e^{NI} - NI - 1}{N} = \frac{NI^2}{2!} + \frac{N^2 I^3}{3!} + \ldots + \frac{N^{i-1} I^i}{i!} + \ldots, \tag{37}$$

it follows that $\overline{T}(cN) > c\overline{T}(N)$, where $c > 1$.

In consequence, for a given flow along the road as a whole, it follows from equations (34), (36) *and* (37) *that the mean delay is reduced by more than r^2 times by introducing $(r-1)$ well-placed islands. As the traffic becomes really heavy, this reduction is much more than r^2.*

These results hold only if the lanes between the islands are sufficiently wide to allow free passage of the vehicles in them, because, in a crowded lane, the negative exponential distribution of vehicle-intervals ceases to hold. If the densities in two different lanes are widely different, then, at the optimum positions derived from the assumptions used here, the islands are so placed that the denser lane becomes too narrow for use in practice. Thus, unless the densities in all lanes are roughly equal, the actual optimum positions of the islands will be more evenly spaced than the theoretical positions, and the actual gain in efficiency will not be so great, as soon as the traffic flow becomes fairly large. However, it is not yet known how considerable is the consequent deviation from optimum efficiency.

For the general case, when the traffic flow is not random, a theorem corresponding to the 'r^2 rule' has not yet been derived, and, here again, it is not known how close to r^2 the actual gain in efficiency would be.

In practice, r is usually equal to 2, corresponding to the presence of a single island. The cases where $r \neq 2$ are not yet of much importance.

4. Pedestrian queuing problems

Using Tanner's notation, let α be the number of pedestrians waiting to cross when a random vehicle passes the crossing-point:

Let β be the number of pedestrians crossing immediately behind a random vehicle.

Let γ be the size of a random group of pedestrians crossing.

Let δ be the number of pedestrians waiting at a random time.

Let ϵ be the size of group with which a random pedestrian crosses.

Let η be the number of pedestrians, at the crossing point, waiting to cross just before a random vehicle-passage at which immediate crossing is possible.

Let $A_r, B_r, C_r, D_r, E_r, H_r$, respectively, be the probabilities that $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ have value r.

Let $A(z), B(z), C(z), D(z), E(z), H(z)$ be the corresponding probability generating functions, i.e. $A(z) = \sum_{r=0}^{\infty} A_r z^r$, etc.†

Let successive intervals between pedestrians be independently distributed, and let the probability that a random interval has length between u and $(u+du)$ be $g(u)\,du \equiv dG(u)$. Let $\int_0^\infty u\,dG(u) = M$. Let $Q_r(v; t)$ be the probability that exactly r pedestrians arrive in an interval of length r, whose beginning is at time t after the arrival of the previous pedestrian.

Let
$$Q(z; v; t) = \sum_{r=0}^{\infty} Q_r(v, t)\, z^r. \tag{38}$$

Let $Q(z; v)$ be the average of $Q(z; v; t)$ over t, i.e.

$$Q(z; v) \equiv \int_0^\infty Q(z; v; t)\, d\Psi(t), \tag{39}$$

where $\Psi(t) \equiv d\Psi(t)$ is the probability that t lies in $(t, t+dt)$, whose value is given by an obvious modification of Lemma 1. $Q(z; v)$ can be considered as the probability generating function for the number of pedestrians arriving in a random interval of length v.

Theorem 3.

(1) $$A(z) = \int_0^\infty Q(z; I+v)\, d_v F(v \mid 0). \tag{40}$$

(2) $$B(z) = k \int_0^\infty Q(z; I+v)\, d_v F(v \mid 0) + h = kA(z) + h. \tag{41}$$

(3) $$C(z) = \frac{kB'(1)z + h[B(z) - B(0)]}{kB'(1) + h[1 - B(0)]}. \tag{42}$$

† The corresponding moment-generating functions are $A(e^t)$, etc.

$$D(z) = m^{-1}\left\{-k^{(1)}+k\left[\int_0^I Q(z;v)\,dv - I\right] + \int_0^I dX(u)\int_0^u dv\int_0^\infty Q(z;u-v+I+w)\,d_w F(w\mid 0)\right\}. \quad (43)$$

$$E(z) = zC'(z)/C'(1) = z[kB'(1)+hB'(z)]/B'(1). \quad (44)$$

$$H(z) = \int_0^\infty Q(z;I+v)\,d_v F(v\mid 0) = A(z), \quad (45)$$

where the value of $Q(z;v)$, as defined by (39), is given by Lemma 5.

LEMMA 5. Let
$$\left.\begin{aligned}\gamma(t) &\equiv \left[\int_t^\infty dG(u)\right]^{-1}\\ \text{and}\quad \Gamma(s;t) &\equiv \gamma(t)\int_t^\infty e^{s(t-u)}\,dG(u).\end{aligned}\right\} \quad (46)$$

Then the Laplace transform of $Q(z;v)$ is

$$L\{s,v;Q(z;v)\} \equiv \int_0^\infty e^{-sv}Q(z;v)\,dv = \frac{1}{Ms}\int_0^s \frac{[1-G(t)][1-\Gamma(s;t)]\,dt}{[1-z\Gamma(s;t)]}. \quad (47)$$

Proof of Lemma 5.

$$Q_0(v;t) = \gamma(t)\int_{v+t}^\infty dG(u),$$

$$Q_r(v;t) = \gamma(t)\int_t^{v+t} Q_{r-1}(v-u+t;t)\,dG(u).$$

Thus
$$\sum_{i=0}^r z^i Q_i(v;t) = Q_0(v;t)+\gamma(t)\int_t^{v+t} z\sum_{i=0}^{r-1} z^i Q_i(v-u+t;t)\,dG(u),$$

so that, if $\sum_{i=0}^\infty z^i Q_i(v;t)$ converges to some function $Q(z;v;t)$, then

$$Q(z;v;t) = Q_0(v;t)+\gamma(t)\int_t^{v+t} zQ(z;v-u+t;t)\,dG(u).$$

In consequence,

$$\int_0^\infty e^{-sv}Q(z;v;t)\,dv = \int_0^\infty e^{-sv}Q_0(v;t)\,dv + z\gamma(t)\int_0^\infty e^{-sv}\,dv\int_t^{v+t} Q(z;v-u+t;t)\,dG(u)$$

$$= \int_0^\infty e^{-sv}Q_0(v;t)\,dv + z\gamma(t)\int_t^\infty e^{-s(u-t)}\,dG(u)\int_0^\infty e^{-sx}Q(z;x;t)\,dx.$$

Thus
$$L\{s,v;Q(z;v;t)\} \equiv \int_0^\infty e^{-sv}Q(z;v;t)\,dv = \frac{\int_0^\infty e^{-sv}Q_0(v;t)\,dv}{1-z\gamma(t)\int_t^\infty e^{s(t-u)}\,dG(u)}.$$

Now
$$\int_0^\infty e^{-sv}\,dv\int_{v+t}^\infty dG(u) = \int_t^\infty dG(u)\int_0^{u-t} e^{-sv}\,dv$$

$$= s^{-1}\int_t^\infty (1-e^{s(t-u)})\,dG(u) \quad (s\neq 0),$$

so that
$$\int_0^\infty e^{-sv}Q_0(v;t)\,dv = \gamma(t)\int_0^\infty e^{-sv}\,dv\int_{v+t}^\infty dG(u) = s^{-1}[1-\Gamma(s;t)].$$

In consequence,
$$L\{s, v; Q(z; v; t)\} = \frac{1-\Gamma(s; t)}{s[1-z\Gamma(s; t)]}. \tag{48}$$

By (39),
$$L\{s, v; Q(z; v)\} = \int_0^\infty L\{s, v; Q(z; v; t)\}\, d\Psi(t), \tag{49}$$

and, by an obvious modification of Lemma 1, the distribution of t is

$$\psi(t)\, dt \equiv d\Psi(t) = M^{-1}\int_t^\infty dG(u) = M^{-1}[1-G(t)]. \tag{50}$$

The lemma follows immediately, by applying (48) and (50) to (49).

Proof of Theorem 3. Consider an arbitrary vehicle-passage instant, at time 0, say. Consider the vehicle intervals in the period $(-\infty, 0)$. Let $(-v', -v)$ be the latest of these intervals to have length at least I. Then the queue at time 0 consists of those pedestrians who arrive during the period $(-v-I, 0)$. As v is also equal to the delay to a pedestrian arriving at a random vehicle-passage,

$$A(z) = \int_0^\infty Q(z; I+v)\, d_v F(v|\, 0),$$

so that part (1) of the theorem has been proved.

Since this proof is valid, irrespective of whether or not crossing is possible at time 0, part (6) of the theorem follows immediately.

With probability k, immediate crossing is possible at a random vehicle-passage, in which case the contribution to $B(z)$ is $H(z)$. With probability h, immediate crossing is not possible, in which case the contribution to $B(z)$ is 1. Thus

$$B(z) = kH(z) + h,$$

which, by part (6), yields part (2) of the theorem.

To prove part (4), consider a random time, and let the next vehicle-passage be v ahead, and the previous vehicle-passage $(u-v)$ behind. Then the probability that u lies in $(u, u+du)$ and v lies in $(v, v+dv)$ is, by (13) and (14), $m^{-1}dX(u)\, dv$, if $v \leqslant u$; 0, if $v > u$. If $v \geqslant I$, the probability-generating function, for given u, v, is unity. If $v < I$, $u \geqslant I$, it is $Q(z; I-v)$. If $v < I$, $u < I$, it is

$$\int_0^\infty Q(z; u-v+I+w)\, d_w F(w|\, 0),$$

as can be seen by using an argument similar to that used in the proof of Part (1). Thus

$$\begin{aligned} mD(z) = &\int_I^\infty dX(u)\int_I^u dv + \int_I^\infty dX(u)\int_0^I Q(z; I-v)\, dv \\ &+ \int_0^I dX(u)\int_0^u du\int_0^\infty Q(z; u-v+I+w)\, d_w F(w|\, 0), \end{aligned}$$

from which part (4) of the theorem follows directly.

To prove part (3), use the arguments of Tanner (1951, p. 387), which, for the general case, give
$$C(z) = \kappa[B(z) - B(0)] + \mu z,$$
where κ and μ are given by

$$1 = \kappa[1-B(0)] + \mu, \quad h = \frac{\kappa B'(1)}{\kappa B'(1) + \mu}.$$

This, after a little calculation, gives the required result.

As a proportion C_r of groups cross with r pedestrians in them, it follows that a proportion $rC_r \Big/ \sum_{r=1}^{\infty} rC_r$ of pedestrians cross in groups of size r, which leads almost immediately to part (5) of the theorem.

Theorem 3 holds for very general conditions, but two special cases are worth mentioning, although the actual results for them will not be stated here. First, the pedestrians may arrive at random, i.e. $g(u) = n e^{-nu}$. This case is considered by Mayne (1951, § 3), though not in much detail. Secondly, the vehicles may also arrive at random, i.e. $x(u) = N e^{-Nu}$. This case is discussed in considerable detail by Tanner (1951, § 3). The particular results for these cases are obtained by inserting the appropriate functions in the formulae of Theorem 3.

In his formulae, which were calculated only for pedestrians arriving singly, Tanner introduces the quantity $v(z) = p(e^t - 1) = p(z-1)$. Using the time-scale $I = 1$, suppose that p_r groups of r pedestrians arrive per unit time on the average ($r > 0$), instead of the case where r takes only the value unity. Then it can be shown that these formulae still hold, where, now,

$$\nu(z) = \sum_{r=1}^{\infty} p_r(z^r - 1). \tag{51}$$

Thus z, in the original formulae, is here to be replaced by

$$\left[1 + \frac{1}{p}\sum_{r=1}^{\infty} p_r(z-1)\right] \quad \left(p = \sum_{r=1}^{\infty} p_r\right).$$

Let $\mu(z) = \nu(z)I^{-1}$. Thus, for pedestrians arriving singly, $\mu(z) = n(z-1)$, since $p = nI$, and, if n_r pedestrians arrive per unit time, $p_r = n_r I$, $n = \sum_{r=1}^{\infty} n_r$, so that

$$\mu(z) = \sum_{r=1}^{\infty} n_r(z^r - 1). \tag{52}$$

Let $$\mu'(1) = \left[\frac{d}{dz}\mu(z)\right]_{z=1}, \quad \mu''(1) = \left[\frac{d^2}{dz^2}\mu(z)\right]_{z=1}.$$

For $g(u) = n e^{-nu}$, it is found that $\gamma(t) = e^{nt}$, $\Gamma(s; t) = n(s+n)^{-1}$. Applying Lemma 5,

$$L(s, v; Q(s; v)) = [s - n(z-1)]^{-1}$$

so that
$$Q(s; v) = e^{n(z-1)v} = e^{\mu(z)v}. \tag{53}$$

Equation (53) still holds when $\mu(z)$ is given by the general formula (52).

Thus, when pedestrians, or groups of pedestrians, reach the crossing-point in a random flow, the results of Theorem 3 can be simplified as follows, using (53).

THEOREM 4. If the pedestrians arrive, so that groups of them of size r reach the crossing-point in a random flow at the rate of n_r per unit time (r a positive integer), then, writing μ, μ' for $\mu(z), \mu'(z)$,

(1) $$A(z) = k e^{I\mu} j(-\mu), \tag{54}$$

(5) $$B(z) = k^2 e^{I\mu} j(-\mu) + h, \tag{55}$$

(3) $$C(z) = \frac{\mu'(1)[Ik - h^{(1)}]z + hk[e^{I\mu} j(-\mu) - e^{-In} j(n)]}{\mu'(1)[Ik - h^{(1)}] + h[1 - k e^{-In} j(n)]}, \tag{56}$$

(4) $$D(z) = \frac{1}{m\mu}\{-\mu k^{(1)} + k[e^{I\mu} j(-\mu)[h(-\mu) - h] + e^{I\mu} - I\mu - 1]\}, \tag{57}$$

(5) $$E(z) = \frac{z\{k\mu'(1)[Ik - h^{(1)}] + k^2 h\mu' e^{I\mu}[Ij(-\mu) - j'(-\mu)]\}}{\mu'(1)[Ik - h^{(1)}]}, \tag{58}$$

(6) $$H(z) = k e^{I\mu} j(-\mu) = A(z). \tag{59}$$

Theorem 4 is proved directly by applying (53) to Theorem 3.† An immediate consequence of it is

† Equations (3·1·6) to (3·1·8), (3·1·12), (3·2·1), (3·2·2) and (3·2·4) to (3·2·7) in Mayne (1951) are incorrect. The correct forms of equations (3·1·12), (3·2·1), (3·2·2) and (3·2·6) are given in equations (47), (59), (55) and (65) of this paper, respectively.

THEOREM 5.

(1) Mean $(\alpha) = \mu'(1)[I - h^{(1)}k^{-1}]$, (60)

(2) Mean $(\beta) = \mu'(1)[Ik - h^{(1)}]$, (61)

(3) Mean $(\gamma) = \dfrac{\mu'(1)[Ik - h^{(1)}]}{k\{\mu'(1)[Ik - h^{(1)}] + h[1 - k\,e^{-In}j(n)]\}}$, (62)

(4) Mean $(\delta) = \dfrac{\mu'(1)}{2m}[I^2k - 2Ih^{(1)} + h^{(2)} + 2h^{(1)2}k^{-1}]$, (63)

(5) Mean $(\epsilon) = 1 + \dfrac{h\mu''(1)}{\mu'(1)} + \dfrac{h\mu'(1)}{[Ik - h^{(1)}]}[I^2k - 2Ih^{(1)} + h^{(2)} + 2h^{(1)2}k^{-1}]$, (64)

(6) Mean $(\eta) = \mu'(1)[1 - h^{(1)}k^{-1}]$. (65)

5. DETERMINATION OF THE GROUP-SIZE DISTRIBUTION ON AN ISLAND BETWEEN TWO LANES OF TRAFFIC

For simplicity, suppose that all pedestrians using the island are crossing in a single direction. (If crossing is possible in both directions, the calculations necessary are much more complicated.) Let the distributions of successive vehicle-intervals in the first, second lanes, respectively, to be crossed by the pedestrians, be denoted by $dX_1(u)$, $dX_2(u)$. Let functions be defined in terms of $X_1(u)$, $X_2(u)$, analogously to those which are defined in terms of $X(u)$, in § 2.

This problem has not been treated in as great detail as the case when the pedestrians are on a pavement, at the edge of the road, but Mayne (1951, § 4) obtained formulae for $H_i(z)$, the analogue of $H(z)$, and $B_i(z)$, the analogue of $B(z)$. Note that

$$B_i(z) = k_2 H_i(z) + h_2. \tag{66}$$

His result for the most general case is too complicated to give here, but the result for the special case when the flow in lane 1 is random is stated without proof, as follows.

THEOREM 6.
$$H_i(z) = \int_0^\infty H_i(y; z)\, dF_2(y - I_2), \tag{67}$$

where

$$\begin{aligned} H_i(y; z) = {} & \{N_1[N_1 e^{-I_1(N_1-\mu)} - \mu]\}^{-1} \\ & \times N_1\mu + (N_1^2 - \mu^2)\exp[I_1(\mu - N_1) + \mu y] + \mu^2 \exp[(\mu - N_1)(I_1 + y)] \\ & + \mu \int_0^y \{N_1 - (N_1 - \mu)\exp[\mu(y + I_1 - u)] + [(N_1 - \mu)e^{I_1\mu} - N_1]\exp[(\mu - N_1)(y - u)]\}\, dF_1(u). \end{aligned}$$

Mayne (1951) also showed that, for this case,

$$\text{Mean}\,(\eta_i) = \mu'(1)[I_2 - h_2^{(1)}k_2^{-1}], \tag{68}$$

which, apart from the suffix 2, is the same as (65).

No correspondingly simple results are obtainable for the variance or the other simple statistics, even when the flow in lane 2 is also random.†

† These results hold under the assumption that pedestrians arrive at random, either singly or in groups of various sizes, so that μ is given by (51).

6. AN EXAMINATION OF SOME EMPIRICAL TRAFFIC DATA

Adams (1936, 1948) provided a considerable amount of data on actual vehicle-passage times, measured by himself, C. C. Judson and D. A. de C. Bellamy, and carried out two statistical tests on it. The object of these tests was to find out whether the flow of traffic in practice was random, and also to determine the conditions under which marked departures from randomness occurred. Here, further statistical tests with the same object are summarized and their most important results are given briefly. All of them can, if desired, be modified so as to test any other given law of traffic flow, in which vehicle-intervals are independently distributed and have the same distribution. The third of these new tests is designed to test the *general* hypothesis that successive vehicle-intervals are independently distributed. For further details, especially for the complete tables of results, see Mayne (1951). For these additional tests, the eleven lanes considered by Adams were used; however, a twelfth lane was added, which had been observed since his report was written. The flows varied from 70 to 710 vehicles per hour.

In his first type of test, Adams compared the observed distributions with those calculated on the assumptions of randomness, for the number of vehicle-intervals larger than t, for various t, and found that they agreed closely.

In his second type of test, the period of observation was divided into successive intervals of fixed size, in practice 10 or 30 sec., and then a count was made of the number of these intervals with r arrivals ($r = 0, 1, 2, \ldots$). Then, if m is the average number of vehicles in the given interval time, the probability of r vehicles arriving during an arbitrary interval is $e^{-m}m^r/r!$, if the traffic stream is in fact random. Thus the expected numbers of intervals containing r arrivals can be compared with the actual numbers, so that a χ^2-test can be applied, or the comparison judged by direct inspection (if agreement is close enough). Adams plotted these observed and expected numbers for three of the traffic lanes considered. Refer to Adams's χ^2 test as Test 1. Some additional, new, tests will now be described, which have been carried out on the eleven lanes considered by Adams, and on one extra lane.

In Test 2, the range $(0, \infty)$ is divided into n ranges R_i, (a_{i-1}, a_i) $(i = 1 \text{ to } n)$, such that $a_0 = 0, a_n = N$, where N is a very large number allowed to tend to infinity. These are assigned so that, if the successive vehicle-intervals u are distributed with probability $X(u)$, then the probability that an arbitrary interval length belongs to any given R_i is $1/n$. The numbers a_i are thus the solutions of the equations

$$\int_0^{a_i} dX(u) = i/n \quad (i = 1 \text{ to } n). \tag{69}$$

Although this method is applicable to general $X(u)$, it is here considered only for

$$X(u) = 1 - e^{-\hat{N}u},$$

as the basic null hypothesis in all these tests is that the traffic flow is random. Here $\hat{N}$ is taken to be the empirical value of the traffic flow, i.e. it is the number of vehicle-arrivals in the period considered, divided by the length of this period. For this case

$$a_i = -\hat{N}^{-1}\log_e(1 - i/n) \quad (i = 1 \text{ to } n). \tag{70}$$

The numbers M_i of vehicle-intervals for the n ranges R_i are counted, and compared with the theoretical numbers M/n, where M is the total number of vehicle-arrivals. The goodness of fit can then be evaluated by a χ^2 test with $(n-2)$ degrees of freedom.

In Test 3 it is determined whether the successive interval lengths are independently distributed. The range $(0, \infty)$ of interval lengths is divided into n ranges R_i (a_{i-1}, a_i), covering the whole of the range, which are assigned so as to have roughly the same number of intervals in each. The numbers M_i of intervals in R_i are counted. The observed numbers M_{ij} of interval pairs, of which the first is in R_i and the second is in R_j, are compared with the corresponding numbers $M_i M_j / M$, which are expected on the hypothesis of independence. The goodness of fit is evaluated by a χ^2 test with $(n-1)^2$ degrees of freedom.

In Test 4 it is determined whether the traffic density varies significantly from one part of the period considered to another. This period is divided into n equal intervals R_i, and the number M_i of vehicle-arrivals counted in R_i is compared with the corresponding number M/n expected on the hypothesis of homogeneity. The goodness of fit is evaluated by a χ^2 test with $(n-1)$ degrees of freedom.

All the tests gave non-significant results, with the following exceptions (bracketed figures give the position of the significance level):

Test 1. Lane no. 11 (0·02 to 0·05).

Test 2. Lanes no. 6 ($< 0{\cdot}001$), 10 (0·001 to 0·01), 11 ($\leqslant 0{\cdot}001$).

Test 3. Lane no. 11 (0·01 to 0·02).

Test 4. Lanes no. 2 (0·01 to 0·02), 11 (0·02 to 0·05).

Considering that twelve lanes were used in each test, the results of Tests 1, 3 and 4 do not decisively reject the hypothesis of random flow in all the lanes, other than no. 11. However, Test 2, which seems much more sensitive than the other tests, gives a deviation from the null hypothesis, which is *extremely* significant for no. 11, and is still very significant for nos. 6 and 10. Thus the conclusion is that all lanes probably have random flow, apart from nos. 6, 10 and 11. Closer examination of the data suggests that the distribution of intervals for these lanes is of form

$$x(u) = N^{c+1} x^c e^{-Nx}/c! \quad (c > 0). \tag{71}$$

For example, c may be taken as 1 or 2. This formula has in fact been suggested by some members of the Road Research Laboratory. The results for no. 11 suggest that, here, the successive intervals are not even independently distributed, and it is probable that the high density of traffic is here causing serious interference.

Greenshields (1946) compared the observed curve of $X(u)$ with the theoretical (negative exponential) curve for several traffic lanes. For traffic flows over about 300 vehicles per hour, he found that the theoretical values were too high for low u, and that, in practice, very few values of u were less than 1·36 sec. In two cases there was an extra mode in the tail of the curve for $x(u)$, as well as the usual mode at a low or zero value of u. Nevertheless, he also concluded that, for flows not too high, the assumption of randomness gives results which are accurate to a reasonable approximation.

I am indebted to the Director and staff of the Road Research Laboratory of the Department of Scientific and Industrial Research for their cooperation in permitting me to read two of their reports, Adams (1948) and Tanner (1951) (before its publication), in supplying me with data of observations on traffic, and in inviting me to visit the Laboratory itself to discuss the problems of road traffic. I am indebted to Mr W. F. Adams for allowing me to discuss the empirical data quoted in his papers. I wish to thank Mr D. G. Kendall and Mr J. C. Tanner for their extensive advice given to me during my research on road traffic, and Mr W. F. Adams, Prof. D. G. Champernowne, Dr F. Garwood, Prof. P. A. P. Moran, Mr P. S. Neall and the referee for several useful suggestions.

REFERENCES

ADAMS, W. F. (1936). *J. Instn Civ. Engrs*, **4**, 121–30.
ADAMS, W. F. (1948). Unpublished Report to the Road Research Laboratory, 42 pp.
GARWOOD, F. (1940). *Suppl. J. R. Statist. Soc.* **7**, 65–77.
GREENSHIELDS, B. D. (1946). *Proc. Highw. Res. Bd, Wash.*, **26**, 377–89.
MALMQUIST, S. (1947). *Ann. Math. Statist.* **18**, 255–64.
MAYNE, A. J. (1951). B.Sc. Thesis, Oxford University, 68 pp. (A copy of this is available in the library of the Royal Statistical Society.)
TANNER, J. C. (1951). *Biometrika*, **38**, 383–92.

BIAS IN THE ESTIMATION OF AUTOCORRELATIONS

By F. H. C. MARRIOTT, *University of Aberdeen*
and J. A. POPE, *Marine Laboratory, Aberdeen*

1. Introduction

1·1. In the analysis of most time series it is customary to estimate the mean and the trend by fitting constants and use the residuals for the estimation of autocorrelations. Although it is quite generally realized that estimates of autocorrelations obtained in this way are liable to bias, the form and full extent of this bias have not been extensively investigated. In a recent note (1953) the authors pointed out that serial correlations of stationary time series are liable to more serious bias than has commonly been supposed. In this paper the bias in serial correlation coefficients derived from certain types of stationary series is considered. In subsequent papers it is hoped to investigate in detail the effect of different methods of trend removal on particular types of series.

1·2. The bias in serial correlation coefficients arises from two sources. In the first place, if the true mean of the series is known the serial correlations will, in general, be biased except when the corresponding autocorrelations are zero, since the serial correlation and the estimated variance are not independently distributed except in special cases. The direction of this bias appears, in some schemes at any rate, to be towards zero.

In practice, the mean frequently has to be estimated from the sample and this introduces a further bias. This second source of bias, which is present even when the autocorrelation is zero, is always negative in a long series, i.e. the expectation of the serial correlation calculated using the estimated mean is always algebraically less, nearer to -1, than the expectation when the known true mean is used.† The two sources of bias may reinforce each other, or may act in opposite directions; they are not independent and cannot be investigated separately.

1·3. Orcutt (1948) appears to have been the first to describe these two sources of bias explicitly. The exact distribution of the serial correlation coefficient (circularly defined) for a series of random normal deviates was derived by Anderson (1942), and the bias in the first serial correlation coefficient when the mean is estimated shown to be $(n-1)^{-1}$. Moran (1948) showed that the bias in all serial correlations of a random series (not necessarily normal) using deviations from the sample mean is $(n-1)^{-1}$ for both circular and non-circular definitions. Sastry (1951) investigated the bias in the serial correlations from certain non-random series but considered only the second source of bias and in consequence under-estimated the total bias.

1·4. A rough method of removing the bias in estimated autocorrelations has been suggested by Quenouille (1949). This method consists of calculating the statistic

$$R = 2r - \tfrac{1}{2}({}_1r + {}_2r),$$

† For, if $|\rho| < 1$ using the definition of (2·04), the correlation using the estimated mean is of the form $(A - C/n)/(B - C/n) + O(n^{-2})$, where $B, C > 0$, and $|A| < B$. This is equal to

$$A/B - C(1 - A/B)/(Bn) + O(n^{-2}),$$

which is less than A/B for large values of n.

where r is the serial correlation coefficient for the whole series and ${}_1r$ and ${}_2r$ are the serial correlations of the first and second halves respectively. This procedure reduces the bias in R to order n^{-2}. However, the use of this adjustment inevitably implies some loss of efficiency. Probably this is not serious as r and ${}_1r+{}_2r$ are very highly correlated. In practial cases, this method is very useful as it requires no assumption about the nature of the series.

1·5. The only method of estimating the extent of the bias in serial correlation coefficients seems to be by evaluation of the sample moments. In the present paper the bias to order n^{-1} in serial correlations is evaluated for two of the simplest types of autocorrelated series. The unweighted moving average series $x_t = \epsilon_t + \epsilon_{t-1}$ and the Markoff series

$$x_t = \rho x_{t-1} + \epsilon_t \quad (|\rho| < 1)$$

are considered in some detail, and the theoretical work is supplemented by sampling investigations on artificially constructed series.

2. DEFINITIONS

2·1. A number of definitions for serial correlation coefficients have been adopted by various writers.

One of these, the circular definition, involves special assumptions, and it will not be considered further in this paper.

When the mean m is known the kth serial correlation coefficient is defined as

$$r_k^* = \frac{\frac{1}{n-k}\sum_{t=1}^{n-k}(x_t-m)(x_{t+k}-m)}{\frac{1}{n}\sum_{t=1}^{n}(x_t-m)^2}. \tag{2·01}$$

When the mean is estimated from the data, three definitions will be considered:

$$r_k^{(1)} = \frac{\sum_{t=1}^{n-k} x_t x_{t+k} - \frac{1}{n-k}\left(\sum_{t=1}^{n-k} x_t\right)\left(\sum_{t=1}^{n-k} x_{t+k}\right)}{\left\{\sum_{t=1}^{n-k} x_t^2 - \frac{1}{n-k}\left(\sum_{t=1}^{n-k} x_t\right)^2\right\}^{\frac{1}{2}}\left\{\sum_{t=1}^{n-k} x_{t+k}^2 - \frac{1}{n-k}\left(\sum_{t=1}^{n-k} x_{t+k}\right)^2\right\}^{\frac{1}{2}}}, \tag{2·02}$$

$$r_k^{(2)} = \frac{\frac{1}{n-k}\sum_{t=1}^{n-k} x_t x_{t+k} - \frac{1}{n^2}\left(\sum_{t=1}^{n} x_t\right)^2}{\frac{1}{n}\sum_{t=1}^{n} x_t^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n} x_t\right)^2}, \tag{2·03}$$

$$r_k^{(3)} = \frac{\frac{1}{n-k}\sum_{t=1}^{n-k} x_t x_{t+k} - \frac{1}{(n-k)^2}\left(\sum_{t=1}^{n-k} x_t\right)\left(\sum_{t=1}^{n-k} x_{t+k}\right)}{\frac{1}{n}\sum_{t=1}^{n} x_t^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n} x_t\right)^2}. \tag{2·04}$$

The first of these definitions requires the calculation of separate estimates of variance for the two sets of observations being correlated, and is rather cumbersome to use in practice especially when a large number of serial correlations are required. The correction for the mean in the numerator of the second definition (which was considered by Sastry) is not strictly appropriate, and in sampling investigations carried out by the authors $r_k^{(2)}$ was found to give occasional values considerably discordant with the other definitions, especially

in short series. This will obviously happen when the end terms differ appreciably from the mean in the same direction. The definition (2·04) employs a sensible correction for the mean in the serial covariance and the same estimate of variance, based on the whole series, for all values of k and appears to be the most suitable of the above definitions. Attention will therefore be confined to $r^{(3)}$ in the rest of this paper when dealing with serial correlations involving an estimated mean.

2·2. It will be evident that the definitions (2·02)–(2·04) differ only by terms of order n^{-2}, and sampling investigations have shown that there is little to choose between them from the point of view of either their biases or their variances.

3. Moments of the serial correlation coefficient in samples from the moving average scheme $x_t = \epsilon_t + \epsilon_{t-1}$†

3·1. In this and succeeding sections the superscript (3) in $r_k^{(3)}$ will be dropped for convenience.

The second-order unweighted moving average scheme is defined as $x_t = \epsilon_t + \epsilon_{t-1}$. In the subsequent sections ϵ_t will be taken as a random normally distributed variate with zero mean and unit variance. It then follows that $\mathscr{E}(x_t) = 0$ and $\mathscr{E}(x_t^2) = 2$.

Writing $r_k = N_k/D$, where

$$N_k = \frac{1}{n-k}\sum_{t=1}^{n-k} x_t x_{t+k} - \frac{1}{(n-k)^2}\left(\sum_{t=1}^{n-k} x_t\right)\left(\sum_{t=1}^{n-k} x_{t+k}\right), \tag{3·01}$$

$$D = \frac{1}{n}\sum_{t=1}^{n} x_t^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n} x_t\right)^2 \tag{3·02}$$

and $\mathscr{E}(N_k) = \nu_k$, $\mathscr{E}(D) = \delta$ we have, to order n^{-1},

$$\mathscr{E}(r_k) = \frac{\nu_k}{\delta}\left\{1 - \frac{\operatorname{cov}(N_k D)}{\nu_k \delta} + \frac{\operatorname{var}(D)}{\delta^2}\right\}. \tag{3·03}$$

The evaluation of (3·03) needs to be carried out separately for $k = 1$, 2 and $k \geqslant 3$.

A little consideration shows that, to order n^{-1}, both $\operatorname{cov}(N_k D)$ and $\operatorname{var}(D)$ are unaffected by estimation of the mean. That is, if

$$N_k' = \frac{1}{n-k}\sum_{t=1}^{n-k} x_t x_{t+k},$$

$$D' = \frac{1}{n}\sum_{t=1}^{n} x_t^2,$$

then

$$\operatorname{cov}(N_k D) = \operatorname{cov}(N_k' D') \text{ to order } n^{-1}, \tag{3·04}$$

$$\operatorname{var}(D) = \operatorname{var}(D') \text{ to order } n^{-1}. \tag{3·05}$$

3·2. For $k = 1$, noting that $\mathscr{E}(x_t x_{t+j}) = 0$ for $j \geqslant 2$, it follows by straightforward algebra that

$$\nu_1 = \frac{n^2 - 6n + 9}{(n-1)^2}, \tag{3·06}$$

$$\delta = \frac{2(n^2 - 2n + 1)}{n^2}. \tag{3·07}$$

† While this work was being prepared for publication the authors learned that general results for the first sample moment of serial correlation coefficients, including the formulae of §§ 3·1 and 4·1 as special cases, have been derived by Prof. M. G. Kendall. See his note following this paper.

For the evaluation of $\mathscr{E}(D'^2)$ the following results, which may be easily proved, are required

$$\mathscr{E}(x_i^4) = 12, \quad \mathscr{E}(x_i^2 x_{i+1}^2) = 6, \quad \mathscr{E}(x_i^2 x_{i+j}^2) = 4 \quad (j \geqslant 2). \tag{3·08}$$

Then
$$\mathscr{E}(D'^2) = \frac{1}{n^2}\mathscr{E}\left\{\left(\sum_{i=1}^{n} x_i^2\right)^2\right\}$$
$$= \frac{1}{n^2}\mathscr{E}\left\{\sum_{i=1}^{n} x_i^4 + 2\sum_{i=1}^{n-1}\sum_{j=1}^{n-i} x_i^2 x_{i+j}^2\right\}$$
$$= \frac{4(n^2+3n-1)}{n^2}, \tag{3·09}$$

and, since $\mathscr{E}(D') = 2$

$$\text{var}\,(D') = \frac{12}{n} - \frac{4}{n^2}. \tag{3·10}$$

Hence
$$\text{var}\,(D) = \frac{12}{n} + O\left(\frac{1}{n^2}\right). \tag{3·11}$$

In a similar manner it is easy to show that

$$\text{cov}\,(N_1 D) = \frac{8}{n} + O\left(\frac{1}{n^2}\right). \tag{3·12}$$

Substituting these results in (3·03) gives

$$\mathscr{E}(r_1) = \frac{1}{2} - \frac{3}{2n} + O\left(\frac{1}{n^2}\right). \tag{3·13}$$

If the true mean is known to be zero the corresponding result is

$$\mathscr{E}(r_1^*) = \frac{1}{2} - \frac{1}{2n} + O\left(\frac{1}{n^2}\right). \tag{3·14}$$

3·3. The results for $k = 2$ and $k \geqslant 3$ are obtained in the same manner, and are found to be

$$\mathscr{E}(r_2) = -\frac{5}{2n} + O\left(\frac{1}{n^2}\right), \tag{3·15}$$

$$\mathscr{E}(r_k) = -\frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad (k \geqslant 3). \tag{3·16}$$

When the true mean is known to be zero the exact result

$$\mathscr{E}(r_k^*) = 0 \quad (k \geqslant 2)$$

is, of course, obtained.

3·4. Expressions for the variance of r_k may be derived using the result, correct to order n^{-1},

$$\text{var}\,(r_k) = \frac{\nu_k^2}{\delta^2}\left\{\frac{\text{var}\,(N_k)}{\nu_k^2} - \frac{2\,\text{cov}\,(N_k D)}{\nu_k \delta} + \frac{\text{var}\,(D)}{\delta^2}\right\}. \tag{3·17}$$

In this case the leading term in the expansion of var (r_k) is the same for all $k \geqslant 2$, and it is therefore only necessary to evaluate (3·17) separately for $k = 1$ and $k \geqslant 2$. Using the fact that var (N_k) = var (N_k') to order n^{-1} some straightforward algebra leads to the results

$$\text{var}\,(r_1) = \frac{1}{2n} + O\left(\frac{1}{n^2}\right) = \text{var}\,(r_1^*), \tag{3·18}$$

$$\text{var}\,(r_k) = \frac{3}{2n} + O\left(\frac{1}{n^2}\right) = \text{var}\,(r_k^*) \quad (k \geqslant 2). \tag{3·19}$$

4. Moments of the serial correlation coefficient in samples from the Markoff scheme $x_t = \rho x_{t-1} + \epsilon_t$

4·1. The mean and variance of r_k may be evaluated to order n^{-1} using formulae (3·03) and (3·17). It will be assumed that $|\rho| < 1$ and that ϵ_t is again a random normal variate with zero mean and unit variance.

A general formula for $\mathscr{E}(r_k)$ may be derived which is valid for all positive values of k. Since $\mathscr{E}(x_t x_{t+j}) = \rho^j \operatorname{var}(x_t)$, where $\operatorname{var}(x_t) = 1/(1-\rho^2)$

$$\nu_k = \frac{\rho^k}{(1-\rho^2)} - \frac{1}{n(1-\rho)^2} + O\left(\frac{1}{n^2}\right), \tag{4·01}$$

$$\delta = \frac{1}{(1-\rho^2)} - \frac{1}{n(1-\rho)^2} + O\left(\frac{1}{n^2}\right). \tag{4·02}$$

In order to evaluate $\operatorname{var}(D)$, etc., the general result

$$\mathscr{E}(x_t x_{t+k} x_{t+k+l} x_{t+k+l+m}) = \frac{\rho^{k+m}(1+2\rho^{2l})}{(1-\rho^2)^2} \tag{4·03}$$

is necessary. Then

$$\begin{aligned}\mathscr{E}(D'^2) &= \frac{1}{n^2}\mathscr{E}\left\{\left(\sum_{t=1}^{n} x_t^2\right)^2\right\} \\ &= \frac{1}{n^2}\left\{\frac{3n}{(1-\rho^2)^2} + 2\sum_{i=1}^{n-1}(n-i)\frac{(1+2\rho^{2i})}{(1-\rho^2)^2}\right\} \\ &= \frac{1}{(1-\rho^2)^2} + \frac{2(1+\rho^2)}{n(1-\rho^2)^3} + O\left(\frac{1}{n^2}\right),\end{aligned}$$

$$\operatorname{var}(D') = \frac{2(1+\rho^2)}{n(1-\rho^2)^3} + O\left(\frac{1}{n^2}\right). \tag{4·04}$$

Again
$$\begin{aligned}\mathscr{E}(N'_k D') &= \frac{1}{n(n-k)}\mathscr{E}\left\{\left(\sum_{t=1}^{n-k} x_t x_{t+k}\right)\left(\sum_{t=1}^{n} x_t^2\right)\right\} \\ &= \frac{1}{n(n-k)}\left\{2(n-k)\mathscr{E}(x_t^2 x_{t+k}) + (n-k)\sum_{j=1}^{k-1}\mathscr{E}(x_t x_{t+j}^2 x_{t+k})\right. \\ &\qquad \left. + 2\sum_{j=1}^{n-k-1}(n-k-j)\mathscr{E}(x_t x_{t+k} x_{t+k+j}^2)\right\} \\ &= \left(1+\frac{k}{n}\right)\left\{\frac{\rho^k}{(1-\rho^2)^2} + \frac{(k+2)\rho^k}{n(1-\rho^2)^2} + \frac{4\rho^{k+2}}{n(1-\rho^2)^3}\right\} + O\left(\frac{1}{n^2}\right) \\ &= \frac{\rho^k}{(1-\rho^2)^2} + \frac{2\rho^k\{(k+1)-(k-1)\rho^2\}}{n(1-\rho^2)^3} + O\left(\frac{1}{n^2}\right),\end{aligned} \tag{4·05}$$

$$\operatorname{cov}(N'_k D') = \frac{2\rho^k\{(k+1)-(k-1)\rho^2\}}{n(1-\rho^2)^3} + O\left(\frac{1}{n^2}\right). \tag{4·06}$$

Substituting these results in (3·03) gives after some algebraic simplification

$$\mathscr{E}(r_k) = \rho^k - \frac{1}{n}\{(1+\rho)(1+\rho+\rho^2+\ldots+\rho^{k-1}) + 2k\rho^k\} + O\left(\frac{1}{n^2}\right). \tag{4·07}$$

When the true mean is known to be zero the corresponding value for r_k^* is

$$\mathscr{E}(r_k^*) = \rho^k - \frac{2k\rho^k}{n} + O\left(\frac{1}{n^2}\right). \tag{4·08}$$

4·2. An asymptotic formula for var (r_k) to order n^{-1} has already been derived by Bartlett (1946). His result, which has been confirmed by the authors, is

$$\text{var}(r_k) = \frac{1}{n}\left\{\frac{(1+\rho^2)(1-\rho^{2k})}{(1-\rho^2)} - 2k\rho^{2k}\right\} + O\left(\frac{1}{n^2}\right). \tag{4·09}$$

The expression (4·09) holds also, of course, for var (r_k^*).

5. SAMPLING INVESTIGATIONS

5·1. The results derived in §§ 3 and 4 were compared with the sampling distributions of r_1 and r_2 for series of different lengths drawn from artificially constructed stochastic series.

5·2. A moving average series containing 1500 terms in all was constructed according to the formula $x_t = \epsilon_t + \epsilon_{t-1}$. The ϵ_t's which had zero mean and unit variance were read consecutively from tables prepared by Wold (1948). The series was started up with a random deviate ϵ_0, x_1 being then taken as $\epsilon_0 + \epsilon_1$ and so on.

The series was first subdivided into 75 consecutive portions each containing 20 terms and the values of r_1 and r_2 calculated for each sample. The values obtained are given in Table 1. The series was then regrouped into 37 sets of 40 items each and the first and second serial correlations for each of these sets are shown in Table 2. Lastly, the series was divided into 25 sets each containing 60 terms and the first and second serial correlations of these samples calculated. These are shown in Table 3.

The frequency distributions of r_1 and r_2 are presented in Tables 4 and 5, together with their means and variances.

5·3. It will be noted that for each length of series the successive values of r_1 and r_2 were calculated from contiguous portions of the whole series and the values of r_1 and r_2 obtained are accordingly not independent. However, for this particular series, $\rho_k = 0$ for $k \geqslant 2$, and the effect of non-independence between successive values of r_1 and r_2 on the sampling distributions of these correlations given in Tables 4 and 5 should be small.

The mean values for both r_1 and r_2 are seen to be in good agreement with the theoretical formulae. In all cases the difference between the observed and theoretical bias is much less than twice its standard error. In addition, the sampling variances also show good agreement with the theoretical values, none of the observed differences being significant as judged by the χ^2 test ($\chi^2 = fs^2/\sigma^2$, f = degrees of freedom for s^2).

5·4. The sampling distributions of r_1 and r_2 were also obtained for samples of size 20, 40 and 60 from an artificially constructed Markoff series with $\rho = 0{\cdot}4$. In order to provide a reasonable number of values for the study of these distributions in samples of size 60 a total of 2100 terms were included in the artificial series. The series was started up with a random normal variate with mean zero and variance $\{1-(0{\cdot}4)^2\}^{-1}$, the random terms thereafter being read consecutively from the tables given by Wold (1948).

The first 10 terms of the series were discarded and samples of size 20, 40 and 60 drawn consecutively, 10 terms being omitted between each sample. In all 30 samples of size 60, 40 samples of size 40, and 60 samples of size 20 were taken. The correlation between the last term of one sample and the first term of the next at a spacing of 10 is of the order of 0·0001. The values of r_1 and r_2 are given in Tables 6, 7 and 8 and the corresponding frequency distributions in Tables 9 and 10. The bias clearly diminishes with increasing n and in all cases the differences between the observed and theoretical biases are much less than twice their

standard errors. The theoretical values given are based on the known population parameters, but the standard errors are estimated from the actual sampling variances.

The observed variances, on the other hand, although decreasing with increasing sample size, are in all cases larger than the theoretical results. Of the six variances only three do not differ significantly from their theoretical values on the basis of the χ^2 test.

5·5. A second Markoff series this time with a parameter 0·8 was constructed. This series contained 2400 terms. It was started up with a random normal variate with zero mean and variance $\{1-(0{\cdot}8)^2\}^{-1}$. The first 20 terms were discarded and samples of size 20, 40 and 60 drawn consecutively omitting 20 terms between each sample. At this spacing the correlation between the last term of one sample and the first of the next is of the order of 0·01. The values of r_1 and r_2 are shown in Tables 11, 12 and 13 and the corresponding frequency distributions in Tables 14 and 15.

The observed means are again in good agreement with expectation, the differences between the observed and the theoretical bias being all less than twice their standard errors.

The agreement between the sample variances and the theoretical values is, on the other hand, disappointing. Except for the value 0·0707 all the observed variances are significantly too high as judged by the χ^2 test.

6. Discussion

6·1. It is clear from the results of the previous sections that estimates of autocorrelations of stationary time series are liable to quite large biases. The sampling investigations indicate that the estimates of bias in the particular series dealt with here should be adequate for most practical purposes. The estimates of variance in the Markoff scheme do not appear satisfactory even for long series.

Although only the bias due to the fitting of a single constant in two simple types of stochastic series has been considered, it seems likely that when more complicated models are dealt with and further constants fitted the bias will generally be at least as serious. In all cases where trend is removed by curve fitting using least squares, however, the bias will tend to zero with increasing sample size provided, of course, the correct order of trend is fitted.

6·2. When moving averages are used for trend elimination, the situation is rather different. This method gives rise to a new correlogram which corresponds to a differencing of the original, and the bias in the serial correlations of the moving average residual series does not in this case tend to zero as n increases indefinitely.

The authors wish to express their thanks to Miss H. McDonald and Miss R. Mair for considerable assistance in carrying out much of the laborious computations.

REFERENCES

Anderson, R. L. (1942). *Ann. Math. Statist.* **13**, 1.
Bartlett, M. S. (1946). *J. R. Statist. Soc. Suppl.* **8**, 27.
Moran, P. A. P. (1948). *Biometrika*, **35**, 225.
Orcutt, G. H. (1948). *J. R. Statist. Soc.* B, **10**, 1.
Pope, J. A. & Marriott, F. H. C. (1953). *Nature, Lond.*, **172**, 778.
Quenouille, M. H. (1949). *J. R. Statist. Soc.* B, **11**, 68.
Sastry, A. S. R. (1951). *Sankhyā*, **11**, 281.
Wold, H. (1948). *Tracts for Computers*, no. xxv. Cambridge University Press.

Table 1. *Values of r_1 and r_2 in samples of size 20 from the moving average scheme $x_t = \epsilon_t + \epsilon_{t-1}$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·446	−0·055	26	0·516	0·016	51	0·325	−0·156
2	0·328	−0·301	27	0·476	−0·050	52	0·411	−0·024
3	0·381	−0·359	28	0·449	−0·064	53	0·507	0·080
4	0·616	0·171	29	0·480	−0·093	54	0·474	0·027
5	0·555	−0·070	30	0·301	−0·096	55	0·478	−0·234
6	0·171	−0·268	31	0·248	−0·213	56	0·530	0·169
7	0·334	0·068	32	0·369	−0·222	57	0·483	−0·128
8	0·296	−0·052	33	0·571	0·235	58	0·434	0·011
9	0·645	0·208	34	0·467	−0·168	59	0·334	−0·290
10	0·621	0·015	35	0·353	−0·691	60	0·292	−0·520
11	0·380	−0·423	36	0·336	−0·510	61	0·446	−0·335
12	0·540	−0·096	37	0·240	−0·470	62	0·647	0·397
13	0·561	0·099	38	0·553	0·256	63	0·334	−0·271
14	0·198	−0·445	39	0·490	−0·132	64	0·305	−0·376
15	0·671	0·363	40	0·469	−0·364	65	0·427	−0·237
16	0·506	−0·010	41	0·120	−0·638	66	0·435	−0·196
17	0·313	−0·194	42	0·478	−0·291	67	0·546	−0·027
18	0·381	−0·247	43	0·260	−0·257	68	0·717	0·188
19	0·359	−0·240	44	0·282	−0·328	69	0·539	−0·052
20	0·215	−0·448	45	0·662	0·094	70	0·545	0·136
21	0·322	−0·386	46	0·598	0·259	71	0·445	−0·232
22	0·516	−0·122	47	0·410	−0·190	72	0·323	−0·462
23	0·309	−0·316	48	0·166	−0·341	73	0·542	−0·065
24	0·280	−0·317	49	0·496	0·053	74	0·229	−0·440
25	0·446	0·042	50	0·284	−0·267	75	0·364	−0·322

Table 2. *Values of r_1 and r_2 in samples of size 40 from the moving average scheme $x_t = \epsilon_t + \epsilon_{t-1}$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·456	−0·154	16	0·416	−0·019	31	0·585	0·157
2	0·489	−0·073	17	0·494	−0·007	32	0·371	−0·182
3	0·363	−0·151	18	0·356	−0·559	33	0·462	−0·112
4	0·341	0·009	19	0·353	−0·189	34	0·634	0·137
5	0·642	0·147	20	0·466	−0·241	35	0·605	0·137
6	0·492	−0·140	21	0·352	−0·330	36	0·380	−0·325
7	0·554	0·077	22	0·309	−0·296	37	0·561	−0·021
8	0·557	0·070	23	0·638	0·145			
9	0·354	−0·156	24	0·286	−0·248			
10	0·331	−0·300	25	0·411	−0·064			
11	0·409	−0·223	26	0·420	0·030			
12	0·285	−0·291	27	0·440	0·007			
13	0·550	0·077	28	0·505	0·039			
14	0·483	0·024	29	0·466	−0·078			
15	0·397	−0·107	30	0·322	−0·391			

Table 3. *Values of r_1 and r_2 in samples of size 60 from the moving average scheme $x_t = \epsilon_t + \epsilon$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·461	−0·144	11	0·476	0·038	21	0·527	0·031
2	0·489	0·038	12	0·379	−0·445	22	0·431	−0·144
3	0·457	0·104	13	0·387	−0·152	23	0·648	0·188
4	0·534	−0·095	14	0·393	−0·325	24	0·400	−0·231
5	0·585	0·148	15	0·561	−0·011	25	0·491	−0·120
6	0·414	−0·092	16	0·499	0·149			
7	0·329	−0·305	17	0·387	−0·091			
8	0·426	−0·164	18	0·524	0·120			
9	0·548	0·092	19	0·495	−0·035			
10	0·422	−0·092	20	0·365	−0·316			

Table 4. *Frequency distributions of r_1 in samples from the moving average scheme $x_t = \epsilon_t + \epsilon$*

r_1	$n = 20$	$n = 40$	$n = 60$
0·70–0·80	1	—	—
0·60–	6	4	1
0·50–	15	6	6
0·40–	20	13	12
0·30–	19	12	6
0·20–	10	2	—
0·10–	4	—	—
Total	75	37	25
Mean	0·421	0·447	0·465
Bias	−0·079	−0·053	−0·035
Predicted bias	−0·075	−0·038	−0·025
S.E. of mean	0·0154	0·0170	0·0155
Variance	0·0178	0·0106	0·0060
Predicted variance	0·0250	0·0125	0·0083

Table 5. *Frequency distribution of r_2 in samples from the moving average scheme $x_t = \epsilon_t + \epsilon_{t-1}$*

r_2	$n = 20$	$n = 40$	$n = 60$
0·30–0·40	2	—	—
0·20–	4	—	—
0·10–	4	5	5
0·00–	10	8	4
−0·10–	13	6	6
−0·20–	8	8	5
−0·30–	13	6	1
−0·40–	11	3	3
−0·50–	6	—	1
−0·60–	2	1	—
−0·70–	2	—	—
Total	75	37	25
Mean	−0·150	−0·097	−0·074
Bias	−0·150	−0·097	−0·074
Predicted bias	−0·125	−0·062	−0·042
S.E. of mean	0·0268	0·0285	0·0334
Variance	0·0539	0·0302	0·0280
Predicted variance	0·0750	0·0375	0·0250

Table 6. *Values of r_1 and r_2 in samples of size 20 from the Markoff scheme $x_t = 0{\cdot}4x_{t-1} + \epsilon_t$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·389	0·199	21	0·203	−0·227	41	0·390	−0·204
2	0·604	0·269	22	−0·187	−0·239	42	0·283	−0·015
3	0·530	0·155	23	0·018	−0·369	43	0·083	−0·159
4	0·477	0·220	24	0·394	0·041	44	0·568	0·368
5	0·151	−0·167	25	0·129	−0·346	45	0·247	−0·119
6	0·630	0·407	26	0·333	−0·284	46	−0·114	−0·024
7	0·595	0·420	27	0·460	0·023	47	0·112	−0·071
8	0·379	0·161	28	0·216	0·048	48	0·514	0·299
9	0·629	0·262	29	0·595	0·141	49	0·254	−0·113
10	0·195	−0·427	30	0·185	0·299	50	0·510	0·049
11	0·628	0·152	31	−0·116	−0·291	51	0·420	−0·133
12	0·256	0·302	32	0·288	−0·302	52	0·178	−0·094
13	0·360	−0·101	33	0·194	−0·095	53	0·038	−0·731
14	0·255	0·087	34	0·187	0·173	54	0·433	0·440
15	−0·265	−0·405	35	0·023	−0·228	55	−0·219	−0·063
16	0·295	−0·389	36	0·529	0·380	56	0·329	−0·416
17	0·248	−0·232	37	0·278	−0·076	57	−0·051	0·034
18	0·260	0·047	38	0·472	0·014	58	0·359	−0·313
19	0·668	0·542	39	0·201	0·207	59	0·247	−0·191
20	0·381	−0·003	40	0·228	0·021	60	0·751	0·471

Table 7. *Values of r_1 and r_2 in samples of size 40 from the Markoff scheme $x_t = 0{\cdot}4x_{t-1} + \epsilon_t$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·282	0·288	16	0·404	−0·083	31	0·422	−0·112
2	0·375	0·132	17	0·177	0·150	32	−0·093	−0·339
3	0·242	−0·143	18	0·201	0·173	33	0·095	−0·106
4	0·660	0·457	19	0·147	−0·178	34	0·429	0·078
5	0·396	−0·003	20	0·556	0·381	35	0·277	−0·313
6	0·503	0·089	21	0·008	−0·097	36	0·695	0·507
7	0·625	0·283	22	0·519	0·403	37	0·053	−0·104
8	0·458	0·164	23	0·447	0·178	38	0·359	−0·043
9	0·139	−0·088	24	0·481	0·202	39	0·360	0·161
10	0·306	−0·090	25	0·343	−0·014	40	0·659	0·380
11	0·388	0·111	26	0·252	0·184			
12	0·311	0·062	27	0·450	0·157			
13	0·139	−0·094	28	0·031	−0·003			
14	0·424	0·003	29	0·666	0·385			
15	0·044	−0·149	30	0·499	0·112			

Table 8. *Values of r_1 and r_2 in samples of size 60 from the Markoff scheme $x_t = 0{\cdot}4x_{t-1} + \epsilon_t$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·285	0·245	11	0·111	−0·343	21	0·589	0·338
2	0·305	−0·121	12	0·405	0·202	22	0·397	−0·043
3	0·598	0·456	13	0·435	0·130	23	0·134	0·066
4	0·560	0·297	14	0·427	0·096	24	0·285	0·046
5	0·558	0·182	15	0·162	−0·024	25	0·215	−0·137
6	0·397	0·149	16	0·494	0·308	26	0·607	0·348
7	0·063	−0·229	17	0·424	0·221	27	0·297	−0·110
8	0·554	0·378	18	0·346	−0·054	28	0·329	0·085
9	0·279	−0·006	19	0·405	0·174	29	0·631	0·284
10	0·271	−0·014	20	0·141	−0·016	30	0·468	0·307

Table 9. *Frequency distributions of r_1 in samples from the Markoff scheme $x_i = 0{\cdot}4x_{i-1} + \epsilon_i$*

r_1	$n=20$	$n=40$	$n=60$
0·70–0·80	1	—	—
0·60–	5	5	2
0·50–	7	3	5
0·40–	5	9	7
0·30–	9	8	5
0·20–	15	5	6
0·10–	8	4	4
0·00–	4	5	1
−0·10–	1	1	—
−0·20–	3	—	—
−0·30–	2	—	—
Total	60	40	30
Mean	0·294	0·343	0·372
Bias	−0·106	−0·057	−0·028
Predicted bias	−0·110	−0·055	−0·037
S.E. of mean	0·0297	0·0315	0·0294
Variance	0·0529	0·0398	0·0258
Predicted variance	0·0420	0·0210	0·0140

Table 10. *Frequency distributions of r_2 in samples from the Markoff scheme $x_i = 0{\cdot}4x_{i-1} + \epsilon_i$*

r_2	$n=20$	$n=40$	$n=60$
0·50–0·60	1	1	—
0·40–	4	2	1
0·30–	3	3	5
0·20–	6	3	5
0·10–	6	10	4
0·00–	9	4	4
−0·10–	8	9	6
−0·20–	7	6	3
−0·30–	7	—	1
−0·40–	5	2	1
−0·50–	3	—	—
−0·60–	—	—	—
−0·70–	—	—	—
−0·80–	1	—	—
Total	60	40	30
Mean	−0·010	0·077	0·107
Bias	−0·170	−0·083	−0·053
Predicted bias	−0·130	−0·065	−0·043
S.E. of mean	0·0347	0·0326	0·0355
Variance	0·0722	0·0424	0·0378
Predicted variance	0·0622	0·0311	0·0207

Table 11. *Values of r_1 and r_2 in samples of size 20 from the Markoff scheme $x_t = 0{\cdot}8x_{t-1} + \epsilon_t$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·566	0·061	21	0·706	0·463	41	0·689	0·431
2	0·773	0·573	22	0·233	−0·140	42	0·753	0·558
3	0·222	0·152	23	0·558	0·422	43	0·933	0·774
4	0·474	0·147	24	·282	−0·110	44	0·613	0·159
5	0·649	0·207	25	0·552	0·356	45	0·855	0·661
6	0·738	0·378	26	0·440	0·116	46	0·774	0·638
7	0·252	0·046	27	0·833	0·684	47	0·620	0·339
8	0·713	0·387	28	0·796	0·566	48	0·406	−0·204
9	0·395	0·025	29	0·505	0·384	49	0·529	0·336
10	0·333	0·101	30	0·238	−0·190	50	0·463	0·093
11	0·769	0·448	31	0·752	0·496	51	0·694	0·479
12	0·289	−0·058	32	0·554	0·169	52	0·643	0·584
13	0·643	0·274	33	0·448	0·051	53	0·756	0·440
14	0·580	0·441	34	0·892	0·679	54	0·646	0·387
15	0·321	0·306	35	0·736	0·651	55	0·902	0·775
16	0·670	0·417	36	0·322	−0·067	56	0·649	0·321
17	0·780	0·570	37	−0·002	−0·065	57	0·822	0·604
18	0·471	0·031	38	0·434	−0·107	58	0·637	0·156
19	0·743	0·584	39	0·254	−0·114	59	0·589	0·290
20	0·617	0·178	40	0·758	0·514	60	0·534	0·023

Table 12. *Values of r_1 and r_2 in samples of size 40 from the Markoff scheme $x_t = 0{\cdot}8x_{t-1} + \epsilon_t$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·702	0·367	16	0·431	0·055	31	0·762	0·507
2	0·620	0·443	17	0·661	0·436	32	0·746	0·614
3	0·811	0·692	18	0·747	0·537	33	0·649	0·383
4	0·755	0·524	19	0·715	0·423	34	0·779	0·643
5	0·789	0·696	20	0·498	0·248	35	0·765	0·547
6	0·564	0·286	21	0·851	0·708	36	0·488	0·192
7	0·582	0·299	22	0·818	0·642	37	0·807	0·619
8	0·412	0·123	23	0·861	0·700	38	0·793	0·543
9	0·710	0·452	24	0·481	0·090	39	0·640	0·199
10	0·676	0·442	25	0·580	0·367	40	0·770	0·602
11	0·822	0·652	26	0·523	0·262			
12	0·570	0·009	27	0·647	0·368			
13	0·780	0·667	28	0·920	0·879			
14	0·683	0·462	29	0·767	0·533			
15	0·725	0·398	30	0·766	0·594			

Table 13. *Values of r_1 and r_2 in samples of size 60 from the Markoff scheme $x_t = 0{\cdot}8x_{t-1} + \epsilon_t$*

Series	r_1	r_2	Series	r_1	r_2	Series	r_1	r_2
1	0·751	0·473	11	0·582	0·354	21	0·886	0·815
2	0·447	0·308	12	0·843	0·773	22	0·772	0·525
3	0·732	0·441	13	0·807	0·697	23	0·831	0·671
4	0·823	0·679	14	0·790	0·596	24	0·752	0·596
5	0·477	0·167	15	0·532	0·299	25	0·609	0·327
6	0·663	0·388	16	0·815	0·657	26	0·837	0·706
7	0·710	0·508	17	0·854	0·675	27	0·565	0·273
8	0·671	0·510	18	0·601	0·355	28	0·830	0·683
9	0·669	0·293	19	0·719	0·539	29	0·725	0·418
10	0·724	0·539	20	0·592	0·314	30	0·827	0·702

Table 14. *Frequency distributions of r_1 in samples from the Markoff scheme $x_i = 0{\cdot}8x_{i-1}+\epsilon$*

r_1	$n=20$	$n=40$	$n=60$
0·90–1·00	2	1	—
0·80–	4	6	10
0·70–	14	16	9
0·60–	12	7	5
0·50–	9	5	4
0·40–	7	5	2
0·30–	4	—	—
0·20–	7	—	—
0·10–	—	—	—
0·00–	—	—	—
−0·10–	1	—	—
Total	60	40	30
Mean	0·580	0·692	0·714
Bias	−0·220	−0·108	−0·086
Predicted bias	−0·170	−0·085	−0·057
S.E. of mean	0·0264	0·0197	0·0217
Variance	0·0419	0·0156	0·0141
Predicted variance	0·0180	0·0090	0·0060

Table 15. *Frequency distributions of r_2 in samples from the Markoff scheme $x_i = 0{\cdot}8x_{i-1}+\epsilon_i$*

r_2	$n=20$	$n=40$	$n=60$
0·80–0·90	—	1	1
0·70–	2	2	3
0·60–	6	9	6
0·50–	7	7	7
0·40–	9	6	3
0·30–	9	5	6
0·20–	3	4	3
0·10–	8	3	1
0·00–	7	3	—
−0·10–	3	—	—
−0·20–	5	—	—
−0·30–	1	—	—
Total	60	40	30
Mean	0·298	0·455	0·509
Bias	−0·342	−0·185	−0·131
Predicted bias	−0·290	−0·145	−0·097
S.E. of mean	0·0343	0·0321	0·0323
Variance	0·0707	0·0412	0·0314
Predicted variance	0·0526	0·0263	0·0175

NOTE ON BIAS IN THE ESTIMATION OF AUTOCORRELATION

By M. G. KENDALL

Division of Research Techniques, London School of Economics

1. It may be convenient to adjoin to the foregoing paper by Dr Marriott and Mr Pope some more general results giving biases to order n^{-1}. These were obtained by Mr B. Gluss, Mr J. Wise and myself. Mr Wise has some methods of greater generality for dealing with serial correlations, but we have not yet embarked on the algebra required to evaluate terms of n^{-2} in any but the simplest cases.

2. Defining a coefficient in terms of three quadratic forms, A, B, C, as

$$r = \frac{A}{(BC)^{\frac{1}{2}}}, \tag{1}$$

we have, to order n^{-1},

$$E(r) = \frac{E(A)}{\{E(B)\,E(C)\}^{\frac{1}{2}}}\left\{1 - \frac{\frac{1}{2}E(ab)}{E(A)\,E(B)} - \frac{\frac{1}{2}E(ac)}{E(A)\,E(C)} + \frac{\frac{1}{4}E(bc)}{E(B)\,E(C)} + \frac{\frac{3}{8}E(b^2)}{E^2(B)} + \frac{\frac{3}{8}E(c^2)}{E^2(C)}\right\}, \tag{2}$$

where a, b, c are the deviations of A, B, C from their respective means.

If we now define

$$B = \frac{1}{(n-k)}\sum_{t=1}^{n-k} x_t^2 - \frac{1}{(n-k)^2}\left\{\sum_{t=1}^{n-k} x_t\right\}^2, \tag{3}$$

$$C = \frac{1}{n-k}\sum_{t=1}^{n-k} x_{t+k}^2 - \frac{1}{(n-k)^2}\left\{\sum_{t=1}^{n-k} x_{t+k}\right\}^2, \tag{4}$$

it is easy to see from considerations of symmetry that

$$E(B) = E(C); \quad E(ab) = E(ac) = \operatorname{cov}(a, b), \quad E(b^2) = E(c^2) = \operatorname{var} b.$$

Also we have

$$E(bc) = E(b^2) + O(n^{-2}), \tag{5}$$

and thus, from (2),

$$E(r_k) = \frac{E(A)}{E(B)} - \frac{\operatorname{cov}(a, b)}{E^2(B)} + \frac{E(A)\operatorname{var} b}{E^3(B)}. \tag{6}$$

The same expression is reached if we put $B = C$ in (1), as is noted by Marriott and Pope.

3. Without loss of generality we let the variance of the series be unity and write $n - k = \nu$. By a straightforward evaluation of expectations we then find

$$E(B) = \frac{1}{\nu}\left(\nu - 1 - \frac{2}{\nu}\sum_{j=1}^{\nu-1}(\nu - j)\rho_j\right), \tag{7}$$

and taking

$$A = \frac{1}{n-k}\sum_{t=1}^{n-k} x_t x_{t+k} - \frac{1}{(n-k)^2}\left(\sum_{t=1}^{n-k} x_t\right)\left(\sum_{t=1}^{n-k} x_{t+k}\right), \tag{8}$$

we find

$$E(A) = \frac{1}{\nu}\left\{\nu\rho_k - \frac{1}{\nu}\sum_{j=0}^{\nu-1}(\nu - j)\rho_{k+j} - \frac{1}{\nu}\sum_{j=1}^{k}(\nu - j)\rho_{k-j} - \frac{1}{\nu}\sum_{j=1}^{r-k-1}(r - k - j)\rho_j\right\}, \quad (k > 0). \tag{9}$$

To order n^{-1} we have, for normal variation,

$$\operatorname{var} b = \frac{2}{\nu}\sum_{j=-\infty}^{\infty}\rho_j^2, \tag{10}$$

$$\operatorname{cov}(a, b) = \frac{2}{\nu}\sum_{j=-\infty}^{\infty}\rho_j\rho_{j+k}. \tag{11}$$

Thus (6) can be evaluated by substituting from (7), (9), (10) and (11).

4. *Case I. Random series*: $\rho_k = 0$ $(k \neq 0)$.

We find

$$E(r_k) = -\frac{1}{\nu}. \tag{12}$$

Case II. $\rho_1 = \rho$, $\rho_k = 0$ $(k \neq 0, 1)$.

We find

$$E(r_1) = \rho + \frac{1}{\nu}(1 + \rho)(4\rho^2 - 2\rho - 1), \tag{13}$$

$$E(r_2) = -\frac{1}{\nu}(1 + 2\rho + 2\rho^2), \tag{14}$$

$$E(r_k) = -\frac{1}{\nu}(1 + 2\rho) \quad (k > 2). \tag{15}$$

Case III. ρ_1, ρ_2, ρ_3 not zero; $\rho_k = 0$ $(k \neq 0, 1, 2, 3)$.
We find

$$E(r_1) = \rho_1 + \frac{1}{\nu}\{-1 - 3\rho_1 - 2\rho_2 - 2\rho_3 - 2\rho_1\rho_2 + 2\rho_1\rho_3 - 4\rho_2\rho_3 + 2\rho_1^2 + 4\rho_1^3 + 4\rho_1\rho_2^2 + 4\rho_3^2\rho_1\}, \tag{16}$$

$$E(r_2) = \rho_2 + \frac{1}{\nu}\{-1 - 2\rho_1 - 3\rho_2 - 2\rho_3 - 2\rho_1^2 - 4\rho_1\rho_3 + 2\rho_1\rho_2 + 2\rho_2^2 + 2\rho_2\rho_3 + 4\rho_1^2\rho_2 + 4\rho_2^3 + 4\rho_2\rho_3^2\}, \tag{17}$$

$$E(r_3) = \rho_3 + \frac{1}{\nu}\{-1 - 2\rho_1 - 2\rho_2 - 3\rho_3 - 4\rho_1\rho_2 + 2\rho_1\rho_3 + 2\rho_2\rho_3 + 2\rho_3^2 + 4\rho_1^2\rho_3 + 4\rho_2^2\rho_3 + 4\rho_3^3\}, \tag{18}$$

$$E(r_k) = -\frac{1}{\nu}(1 + 2\rho_1 + 2\rho_2 + 2\rho_3). \tag{19}$$

Case IV. Markoff scheme.

$$E(r_k) = \rho^k - \left\{\frac{1+\rho}{1-\rho}(1-\rho^k) + 2k\rho^k\right\}, \tag{20}$$

and in particular

$$E(r_1) = \rho - \frac{1+3\rho}{\nu}. \tag{21}$$

Equation (13) with $\rho = \frac{1}{2}$ gives us Marriott and Pope's formula (3·13); equation (14) and (15) likewise give their (3·15) and (3·16). Equations (20) and (21) agree with their (4·07).

Case V. Case I with a circular definition, i.e.

$$r_k = \frac{\frac{1}{n}\sum_{t=1}^{n} x_t x_{t+k} - \frac{1}{n^2}\left(\sum_{t=1}^{n} x_t\right)^2}{\frac{1}{n}\sum_{t=1}^{n} x_t^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n} x_t\right)^2}. \tag{22}$$

We find

$$E(r_k) = -\frac{1}{n} \quad (k \neq 0). \tag{23}$$

Case VI. Case II with a circular definition.

We find

$$E(r_1) = \rho - \frac{1}{n}(1 + 4\rho - 2\rho^2 - 4\rho^3), \tag{24}$$

$$E(r_2) = -\frac{1}{n}(1 + 2\rho + 2\rho^2), \tag{25}$$

$$E(r_k) = -\frac{1}{n}(1 + 2\rho) \quad (k > 2). \tag{26}$$

Case VII. Case III with a circular definition.

$$E(r_1) = E_1(r_1) - \frac{\rho_1}{n}, \tag{27}$$

$$E(r_2) = E_1(r_2) - \frac{2\rho_2}{n}, \tag{28}$$

$$E(r_3) = E_1(r_3) - \frac{3\rho_3}{n}, \tag{29}$$

$$E(r_k) = E_1(r_k) \quad (k > 3), \tag{30}$$

when E_1 refers to the corresponding expectations of case III.

Case VIII. Case IV (Markoff) with a circular definition.

$$E(r_k) = \rho^k - \frac{1}{n}\left\{\frac{1+\rho}{1-\rho}(1-\rho^k) + 3k\rho^k - k\rho^{n-k}\right\}, \tag{31}$$

and in particular, for $k = 1$,

$$E(r_1) = \rho - \frac{1}{n}(1 + 4\rho). \tag{32}$$

5. Expressions such as these are probably satisfactory for values of ρ near to zero, but they are of very doubtful validity for ρ near to unity. For one thing, the distribution of r is, in general, so highly skewed that the use of expectations as criteria of bias is itself open to question. For another, the use of expansions such as those employed in this note, which in general are asymptotic, may be dubious where there is not a rapid fall in successive terms near the beginning. It is not necessarily true that better results would be obtained by evaluating further terms to order n^{-2} or n^{-3}. We are inclined to think that the problem is most likely to yield to a new approach.

TESTS OF HYPOTHESES IN THE LINEAR AUTOREGRESSIVE MODEL

1. NULL HYPOTHESIS DISTRIBUTIONS IN THE YULE SCHEME

BY G. M. JENKINS
University College, London

1. INTRODUCTION

The stochastic approach in time series, namely, that of determining whether a sequence of random variables could have been generated by a given process, is one which has received considerable attention since Yule (1927) fitted an autoregressive series to Wolfer's sunspot numbers.

In the case of the autoregressive model, it has been generally accepted that the 'best' method of fitting is to test for a model of increasing order until an arbitrary number of non-significant results is obtained. Whittle (1952) has discussed how this procedure may be regarded as including all purely non-deterministic stationary processes as alternative hypotheses to the autoregressive model considered.

Up to the present, valid inferences have been hindered by the following considerations:

(1) The sampling distributions of the test statistics considered were not known, thus making it necessary to assume that the null hypothesis distributions are approximated by the analogous forms in the theory of multiple regression.

(2) As a result, large sample tests have been derived by Quenouille (1947) and Whittle (1952). In the case of the former, it has been shown, for example, by Rao & Som (1951) that the test is not satisfactory for short series. It may thus be inferred that the same is true of the latter, since the asymptotic nature of the approximation is the same in both tests.

It is thus proposed to consider the distribution problem systematically, for the present confining the discussion to the testing of hypotheses in the Yule scheme.

In order to determine the distributions of the test criteria (which take the form of partial and multiple serial correlation coefficients) it is necessary to know the joint distribution of the serial correlations of different lags. The only paper which has appeared on this subject is that by Quenouille (1949*a*), who derives an explicit expression for the joint distribution of the first k serial correlation coefficients. The resulting distribution is intractable, and hence an integral approximation is suggested as a k-dimensional analogue to the smoothed form* for $p(r_1)$ proposed by Dixon (1944). It will be shown subsequently that whereas the integral form has correct moments as far as the marginal distribution of r_1 is concerned, other marginal and multivariate moments are incorrect.

* This has the same moments as the exact distribution up to order n and is represented by a single function throughout the range of the variable, whereas the exact distribution is defined by different functional forms for various sections of the range.

2. Scope of present study

The greater part of the paper is devoted to the case where the observations are assumed to be drawn from a population with zero mean. Methods are considered for obtaining the bivariate moments of the joint distribution of the first two circularly defined serial coefficients and of the univariate moments of the first partial serial coefficient. A smoothed joint distribution for $p(r_1, r_2)$, which is the two-dimensional analogue of the Dixon distribution, is constructed from a knowledge of these moments. It is found that the smoothed distribution of the multiple serial correlation coefficient of x_i on x_{i+1} and x_{i+2} is the same as that of the ordinary multiple correlation coefficient based on $n+3$ observations. The analysis is then modified to deal with the case where the serial coefficients are corrected for the sample mean and the theory illustrated by results on artificial series.

It is proposed to deal at a later stage with non-central distributions in the Yule scheme, the extension to higher order schemes and applications to real and artificial series.

3. Maximum-likelihood criteria

The problem may be formalized as follows:

Given a realization $\{x_i\}$ $(i = 1, 2, \ldots, n)$ of a time series T, it is assumed that the theoretical mean of the series is zero and that the underlying model is

$$x_i = \alpha_1 x_{i-1} + \alpha_2 x_{i-2} + \alpha_3 x_{i-3} + \ldots + \alpha_k x_{i-k} + z_i, \tag{3·1}$$

where z_i is a $N(0, \sigma)$ variate and we employ an artificial circular definition $x_{n+i} = x_i$. It is required to test various hypotheses about the parameters $\alpha_1, \alpha_2, \ldots, \alpha_k$.

In the case of the Markoff scheme $\{\alpha_j = 0, j > 1\}$ the problem has been discussed by Dixon (1944), who obtained a smoothed distribution for r_1 agreeing with the exact distribution for the first n moments. If the hypothesis $\{\alpha_j = 0, j > 1\}$ is tenable, the author (1954) has shown that a test of significance for α_1 may be made using the inverse sine transformation.

In the case of a Yule scheme $\{\alpha_j = 0, j > 2\}$ the various hypotheses and the test criteria may be specified in the usual Neyman-Pearson notation:

(1) H_0: $\alpha_1 \neq 0$, $\alpha_2 = 0$; H_1: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$;

$$\lambda^{2/n} = 1 - v^2, \quad \text{where} \quad v = \frac{r_2 - r_1^2}{1 - r_1^2}.$$

(2) H_0: $\alpha_1 = 0$, $\alpha_2 \neq 0$; H_1: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$;

$$\lambda^{2/n} = 1 - u^2, \quad \text{where} \quad u = \frac{r_1(1 - r_2)}{\sqrt{\{(1 - r_1^2)(1 - r_2^2)\}}}.$$

(3) H_0: $\alpha_1 = 0$, $\alpha_2 = 0$; H_1: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$;

$$\lambda^{2/n} = (1 - r_1^2)(1 - v^2) = 1 - T_{20}^2.$$

It is to be noted that whereas $\hat{\alpha}_2 = v$, $\hat{\alpha}_1 = \sqrt{\left(\frac{1 - r_2^2}{1 - r_1^2}\right)} u$, a factor which will prove troublesome when non-central distributions are considered.

Whittle (1952) has shown that the least-square statistics (which apply provided the z's are mutually independent whether they are normally distributed or not) are identical with the maximum-likelihood criteria quoted above.

4. QUENOUILLE'S DISTRIBUTION

Quenouille's integral approximation for the joint distribution of the first k serial correlations (uncorrected for the mean) takes the following form when $k = 2$:

$$p(r_1, r_2) = \frac{1}{\pi} \frac{\Gamma(\frac{1}{2}n+1)\,\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n+\frac{1}{2})\,\Gamma(\frac{1}{2}n-\frac{1}{2})} \left(\frac{R_2}{R_1}\right)^{\frac{1}{2}(n-3)}, \tag{4·1}$$

with the conditions that $R_i \geqslant 0$ $(i = 1, 2)$, where

$$R_1 = \begin{vmatrix} 1 & r_1 \\ r_1 & 1 \end{vmatrix} \quad \text{and} \quad R_2 = \begin{vmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{vmatrix}.$$

The boundary values obtained by equating R_2 to zero are given by

$$\left.\begin{aligned} 2r_1^2 - 1 \leqslant r_2 \leqslant 1, \\ -\sqrt{\{\tfrac{1}{2}(1+r_2)\}} \leqslant r_1 \leqslant \sqrt{\{\tfrac{1}{2}(1+r_2)\}}. \end{aligned}\right\} \tag{4·2}$$

If the joint distribution is integrated between the appropriate limits for r_2, we obtain Dixon's expression for $p(r_1)$, but integration with respect to r_1 does not give $p(r_2)$ which Dixon showed to have the same form as $p(r_1)$.* This is confirmed by an examination of the moments of (4·1).

Transforming to new variables r_1 and $v = (r_2 - r_1^2)(1 - r_1^2)^{-1}$, it may be shown that (4·1) implies that v is independently distributed of r_1 with distribution

$$p(v) = K(1-v^2)^{\frac{1}{2}(n-3)}, \tag{4·3}$$

so that

$$\begin{aligned} \mathscr{E}(r_2) &= \int_{-1}^{+1} \int_{r_1'}^{r_1'} r_2 p(r_1, r_2)\, dr_2 dr_1 \\ &= \int_{-1}^{+1} \int_{-1}^{+1} p(r_1)\, p(v)\, [v(1-r_1^2) + r_1^2]\, dr_1 dv \\ &= \frac{1}{n+2}. \end{aligned}$$

In a similar manner it may be shown that

$$\mathscr{E}(r_2^2) = \frac{n^2+7n+3}{n(n+2)(n+4)}.$$

These expressions are obviously incorrect because $\mathscr{E}(r_2) = 0$ and $\mathscr{E}(r_2^2) = (n+2)^{-1}$. Since the univariate moments are known and are the same for both variables, it is necessary as a first step in the construction of the joint distribution to evaluate the correct bivariate moments.

5. THE BIVARIATE MOMENTS OF $p(r_1, r_2)$

These may be obtained directly by considering the expectations of homogeneous forms in normal variables. This is an extension to the bivariate case of a technique which is due to Moran (1948). Thus

$$\mu'_{st} = \mathscr{E}(r_1^s r_2^t) = \mathscr{E}\left[\frac{(\Sigma x_i x_{i+1})^s\, (\Sigma x_i x_{i+2})^t}{(\Sigma x_i^2)^{s+t}}\right]. \tag{5·1}$$

* It has been drawn to the author's notice that this observation was also made by G. S. Watson (1951) in a mimeographed thesis, North Carolina State College.

This process is made easier by observing that an examination of the n-dimensional space of the x's shows that $p(r_1, r_2)$ is functionally independent of the distribution of $p = \Sigma x_i^2$. (5·1) may thus be evaluated by dividing the expectation of the numerator by that of the denominator.

It may be shown that $\mu'_{11}, \mu'_{12}, \mu'_{31}$ are all zero. As an example of a non-vanishing moment we shall evaluate

$$\mu_{22} = \mathscr{E}(r_1^2 r_2^2) = \frac{\mathscr{E}(q_1^2 q_2^2)}{\mathscr{E}(p^4)},$$

where $p = \Sigma x_i^2, \quad q_1 = \Sigma x_i x_{i+1}, \quad q_2 = \Sigma x_i x_{i+2}$,

$$q_1^2 = \Sigma x_i^2 x_{i+1}^2 + 2\Sigma x_i x_{i+1}^2 x_{i+2} + 2\Sigma x_i x_{i+1} x_{i+2} x_{i+3} + 2 \sum_{\substack{i+1<j \\ j+i+2}}{}' x_i x_{i+1} x_j x_{j+1},$$

the summation Σ' being over distinct values of $i, i+1, j, j+1$, and

$$q_2^2 = \Sigma x_i^2 x_{i+2}^2 + 2\Sigma x_i x_{i+1} x_{i+2} x_{i+3} + 2\Sigma x_i x_{i+2}^2 x_{i+4} + 2 \sum_{\substack{i+1<j \\ j+i+1}}{}' x_i x_{i+2} x_j x_{j+2}.$$

Hence $\mathscr{E}(q_1^2 q_2^2) = \mathscr{E}[\Sigma x_i^4 x_{i+1}^2 x_{i+2}^2 + \Sigma x_i^2 x_{i+1}^4 x_{i+2}^2 + \Sigma x_i^2 x_{i+1}^2 x_{i+2}^4 + \Sigma x_i^2 x_{i+1}^2 x_{i+2}^2 x_{i+3}^2 + \Sigma x_i^2 x_{i+1}^2 x_{i+2}^2 x_{i+3}^2 + 4\Sigma x_i^2 x_{i+1}^2 x_{i+2}^2 x_{i+3}^2 + \Sigma' x_i^2 x_{i+1}^2 x_j^2 x_{j+2}^2 + \text{terms with zero expectations}].$

We may take $\mathscr{E}(x_i^2) = 1$ without any loss of generality so that $\mathscr{E}(x_i^4) = 3$. Since the last summation contains $n(n-4)$ terms, it may be shown that

$$\mathscr{E}(q_1^2 q_2^2) = n(n+12),$$

from which it follows that

$$\mu'_{22} = \frac{n+12}{(n+2)(n+4)(n+6)}. \tag{5·2}$$

The algebra becomes difficult after the first few moments, but the following results were obtained:

$$\left.\begin{aligned} \mu'_{21} &= \frac{2}{(n+2)(n+4)}, \\ \mu'_{41} &= \frac{12}{(n+2)(n+4)(n+6)}. \end{aligned}\right\} \tag{5·3}$$

The method is simplified considerably by working with the joint characteristic function of

$$p' = \frac{1}{2\sigma^2}\Sigma x_i^2, \quad q'_1 = \frac{1}{2\sigma^2}\Sigma x_i x_{i+1}, \quad q'_2 = \frac{1}{2\sigma^2}\Sigma x_i x_{i+2},$$

which is the circulant

$$\phi = \prod_{k=1}^{n} \left\{1 - \theta_0 - \theta_1 \cos\frac{2\pi k}{n} - \theta_2 \cos\frac{4\pi k}{n}\right\}^{-\frac{1}{2}}. \tag{5·4}$$

It follows from the original form of the characteristic function before integrating out that

$$\mu'_{st} = \int_{-\infty}^{0} \left[\int \cdots \int \left[\frac{\partial^{s+t}\phi}{\partial\theta_1^s \partial\theta_2^t}\right]_{\theta_1=0}^{\theta_2=0} (d\theta_0)^{s+t-1}\right] d\theta_0, \tag{5·5}$$

where the expression above is intended to convey that after differentiation and equating to zero, we integrate successively with respect to θ_0 $(s+t)$ times and then evaluate the resulting expression between the limits $-\infty$ and 0 for θ_0.

ϕ may be rewritten in the form

$$\exp\left[-\frac{1}{2}\sum_{k=1}^{n}\log\left\{1-\theta_0-\theta_1\cos\frac{2\pi k}{n}-\theta_2\cos\frac{4\pi k}{n}\right\}\right]=\exp\left[\psi(\theta_0,\theta_1,\theta_2)\right].$$

The following notation will be used:

(i) $L_{st}=\dfrac{\partial^{s+t}\phi}{\partial\theta_1^s\partial\theta_2^t}$, $\quad M_{st}=[L_{st}]_{\substack{\theta_1=0\\ \theta_2=0}}$.

(ii) E_{st} and F_{st} are the corresponding quantities with ϕ replaced by ψ.

(iii) $\cos\dfrac{2\pi k}{n}=\alpha$, $\quad\cos\dfrac{4\pi k}{n}=\beta$, $\quad$ so that $\quad 2\alpha^2=(1+\beta)$.

(iv) $1-\theta_0-\theta_1\alpha-\theta_2\beta=A$, $\quad 1-\theta_0=A_0$.

We shall need to evaluate series of the form $\sum_{k=1}^{n}\cos^s\dfrac{2\pi k}{n}\cos^t\dfrac{4\pi k}{n}$ or $\Sigma\alpha^s\beta^t$ in our notation. It may be proved by induction that

$$\Sigma\alpha^{2s-1}=0, \quad \Sigma\alpha^{2s}=\frac{n(2s-1)(2s-3)\dots 1}{2^s s!}. \tag{5·6}$$

Since $\Sigma\alpha^{i+2}=\frac{1}{2}\Sigma\alpha^i+\frac{1}{2}\Sigma\alpha^i\beta$, it follows that

$$\Sigma\alpha^{2s-1}\beta=0, \quad \Sigma\alpha^{2s}\beta=\frac{n\,2s(2s-1)(2s-3)\dots 1}{2^{s+1}(s+1)!}. \tag{5·7}$$

In analogous fashion, it is seen that

$$\Sigma\alpha^{2s-1}\beta^2=0, \quad \Sigma\alpha^{2s}\beta^2=\frac{4n(s^2+s+1)(2s-1)(2s-3)\dots 1}{2^{s+2}(s+2)!}. \tag{5·8}$$

ϕ may be differentiated successively with respect to θ_1 to give a system of recursive equations, e.g.

$$L_{50}=\phi[E_{50}+5E_{10}E_{40}+10E_{20}E_{30}+15E_{10}E_{20}^2+10E_{10}^2E_{30}+10E_{10}^3E_{20}+E_{10}^5],$$

and generally, if $E_{10}^{t_1}E_{20}^{t_2}\dots E_{k0}^{t_k}$ is a typical term in the expansion of $L_{10}\phi^{-1}$, where $\sum_{i=1}^{k}it_i=t$, then the coefficient of this term will be

$$\frac{t!}{\prod_{i=1}^{k}(i!)^{t_i}(t_i!)}.$$

The following observations may now be made:

(i) Since $\qquad \psi=-\frac{1}{2}\Sigma\log\{1-\theta_0-\theta_1\alpha-\theta_2\beta\}$

it follows that $\qquad E_{r0}=\dfrac{(r-1)!}{2}\Sigma\left(\dfrac{\alpha}{A}\right)^r$. $\tag{5·9}$

(ii) Using the fact that $\Sigma\alpha^{2s-1}\beta^j=0$, it follows that $M_{2s-1,\,j}=0$, which implies that

$$\mathscr{E}(r_1^{2s-1}r_2^j)=0.$$

(iii) The only terms which contribute to $M_{2s,1}$ are those in the expansion of $L_{2s,0}$ which are of even order; thus for M_{41}, the relevant terms in L_{40} are E_{40} and $2E_{20}^2$.

We proceed to illustrate the method by evaluating μ'_{42}. It may be shown that the only terms in L_{41} which lead to non-vanishing series in M_{42} are

$$\phi[E_{41}+6E_{20}E_{21}+3E_{20}^2E_{01}+E_{40}E_{01}],$$

from which it follows that

$$M_{42} = \phi[F_{42} + 6F_{22}F_{20} + 6F_{21}^2 + 3F_{20}^2F_{02} + F_{40}F_{02}],$$

which reduces to

$$M_{42} = A_0^{-\frac{1}{2}n-6}[\Sigma\alpha^4\beta^2 + 6\Sigma\alpha^2\beta^2\Sigma\alpha^2 + 6(\Sigma\alpha^2\beta)^2 + 3(\Sigma\alpha^2)^2\Sigma\beta^2 + \Sigma\alpha^4\Sigma\beta^2].$$

After substituting for the series, this becomes

$$M_{42} = 3n(n+10)(n+28)(1-\theta_0)^{-\frac{1}{2}n-6}/2^6,$$

so that

$$\mu'_{42} = \frac{3(n+28)}{(n+2)(n+4)(n+6)(n+8)}.$$

The moments obtained by this method are exact, but it suffers from the disadvantage that the number of terms in the expansions increases rapidly with the order of the moment.

A large number of examples was worked out, and it appeared from the form of the moments up to order 12 that the following relationships hold:

$$\mu'_{2s,1} = \frac{2s(2s-1)(2s-3)\dots 1}{(n+2)(n+4)\dots(n+2s+2)},$$

$$\mu'_{2s,2} = \frac{[(n+3)+(2s+1)^2](2s-1)(2s-3)\dots 1}{(n+2)(n+4)\dots(n+2s+4)}.$$

The above method will be modified to deal with the case where the serials are corrected for the mean in § 9.

6. The moments of the first partial serial correlation

In this section, three different methods of obtaining the moments of v are discussed. This was found necessary since each brings out different features of the distribution.

Method 1. It is not possible to determine the moments of v by performing operations on the exact characteristic function, but this difficulty is overcome by working with the smoothed form given by Dixon (1944). The latter is extremely cumbersome and it was not found practical to obtain the moments of higher order than 4.

For $\mathscr{E}(v)$ it is necessary

(i) to evaluate $$\left[\frac{\partial^2\phi}{\partial\theta_0\partial\theta_2} - \frac{\partial^2\phi}{\partial\theta_1^2}\right]_{\theta_2=0},$$

(ii) to integrate with respect to $\frac{1}{2}\{1-(\theta_0-\theta_1)\} = x$ and $\{1-(\theta_0+\theta_1)\} = y$ between the limits $\frac{1}{2}$ and ∞ for both variables.

It was found that

$$\mathscr{E}(v) = -\frac{1}{n+1}, \quad \mathscr{E}(v^2) = \frac{1}{n+1},$$

$$\mathscr{E}(v^3) = -\frac{3}{(n+1)(n+3)}, \quad \mathscr{E}(v^4) = \frac{3}{(n+1)(n+3)}.$$

The pattern in the results is striking because the odd moments are the same as the even moments with a change in sign.

It is to be noted that Dixon obtained the first two moments of the criterion ${}_0\lambda_{2,1}^{2/n}$ which is equivalent to $1-v^2$, and these results are in agreement with the even moments given above.

Method 2. It is now assumed that r_1 and v are independent and the assumption is justified at a later stage. This method of approach is extremely interesting in that it leads directly to a smoothed joint distribution for the first two serials with correct moments up to order n.

The assumption of independence in the case of the smoothed distributions will be justified by showing that:

(1) The resulting moments of v obtained by this method are the same as those derived by method 1.

(2) We may transform the joint distribution of r_1 and v to that of r_1 and r_2 and show that the moments of the latter distribution are correct up to order n.

There is some *a priori* evidence for assuming that the two variates are independently distributed, since the condition $R_2 \geqslant 0$ which must be satisfied by any sequence of random variables is equivalent to

$$(1-r_1^2)(1-v^2) \geqslant 0.$$

It follows that the limits of variation of r_1 and v are -1 and $+1$.

It will be seen that the procedure is essentially one of constructing a distribution $p(r_1, r_2)$ which has the correct marginal moments—it will be shown subsequently that this leads to correct bivariate moments in addition.

Since
$$\mathscr{E}(r_2) = \int_{-1}^{+1}\int_{-1}^{+1} p(r_1)\,p(v)\,[v(1-r_1^2)+r_1^2]\,dr_1\,dv,$$

it follows that
$$0 = \mathscr{E}(v)\frac{n+1}{n+2} + \frac{1}{n+2},$$

so that
$$\mathscr{E}(v) = -\frac{1}{n+1}.$$

The smoothed distribution of v may then be constructed by evaluating $\mathscr{E}(r_2^k)$ for different k and then solving the resulting equations for the moments of v. It is simpler to proceed as follows:

Since $1-v = \dfrac{1-r_2}{1-r_1^2}$, the assumption of independence leads to

$$\mathscr{E}(1-v)^k\,\mathscr{E}(1-r_1^2)^k = \mathscr{E}(1-r_2)^k,$$

giving a system of equations in the form

$$\mathscr{E}(1-v)\frac{n+1}{n+2} = 1,$$

$$\mathscr{E}(1-v)^2\frac{(n+1)(n+3)}{(n+2)(n+4)} = \frac{n+3}{n+2},$$

$$\mathscr{E}(1-v)^3\frac{(n+1)(n+3)(n+5)}{(n+2)(n+4)(n+6)} = \frac{n+5}{n+2}, \quad \text{etc.}$$

These have solutions

$$\left.\begin{aligned} \mu'_{2k} &= \frac{(2k-1)(2k-3)\ldots 1}{(n+1)(n+3)\ldots(n+2k-1)}, \\ \mu'_{2k-1} &= -\mu'_{2k}, \end{aligned}\right\} \tag{6·1}$$

and it may be verified that these are the moments of the type I distribution

$$p(v) = \frac{1}{2^{p+q}B(p,q)}(1+v)^{q-1}(1-v)^{p-1},$$

with $p = \frac{1}{2}n + 1$ and $q = \frac{1}{2}n$. This may be rewritten in the form

$$p(v) = \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n)}(1-v^2)^{\frac{1}{2}(n-2)}(1-v). \tag{6·2}$$

This is to be compared with Quenouille's smoothed form for the distribution, namely,

$$\phi(v) = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n - \frac{1}{2})}(1-v^2)^{\frac{1}{2}(n-3)}.$$

It is precisely the fact that $\phi(v)$ is symmetrical which leads to incorrect moments for $p(r_1, r_2)$.

We may now write the joint distribution of r_1 and v in the form

$$p(r_1, v) = \frac{n}{2\pi}(1-r_1^2)^{\frac{1}{2}(n-1)}(1-v^2)^{\frac{1}{2}(n-2)}(1-v),$$

and a transformation yields

$$p(r_1, r_2) = \frac{n}{2\pi}\left(\frac{R_2}{R_1}\right)^{\frac{1}{2}(n-2)}\frac{(1-r_2)}{R_1^2}. \tag{6·3}$$

In § 7 the above distribution will be identified with a smoothed form with correct moments up to order n.

The joint distribution of r_1 and $x = v^2$, where

$$p(r_1, x) = \frac{n}{2\pi}(1-r_1^2)^{\frac{1}{2}(n-1)}x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}(n-2)},$$

may now be transformed to the joint distribution of r_1 and T_{20}^2, where

$$(1-T_{20}^2) = (1-r_1^2)(1-v^2).$$

$p(T_{20}^2)$ may then be obtained by integration of the latter distribution with respect to r_1, using the fact that r_1 and T_{20}^2 vary subject to the condition that $0 \leqslant r_1^2 \leqslant T_{20}^2 \leqslant 1$. Thus

$$p(T_{20}^2) = \tfrac{1}{2}n(1-T_{20}^2)^{\frac{1}{2}(n-2)}, \tag{6·4}$$

which is the same as the distribution of the square of the corresponding ordinary multiple correlation coefficient based on $n+3$ observations.

Dixon (1944) derived the first two moments of ${}_0\lambda_2^{2/n} = (1-T_{20}^2)$ from the smoothed characteristic function by the tedious differentiation process mentioned earlier and obtained

$$\mathscr{E}(x) = \frac{n}{n+2},$$

$$\mathscr{E}(x^2) = \frac{n}{n+4},$$

where $x = {}_0\lambda_2^{2/n}$. These results are in agreement with (6·4) and can be obtained directly since, e.g.

$$\mathscr{E}(1-T_{20}^2) = \mathscr{E}(1-r_1^2)\,\mathscr{E}(1-v^2) = \frac{n+1}{n+2}\frac{n}{n+1}.$$

This must be regarded as strong evidence for the assumption of independence of r_1 and v (or perhaps quasi-independence since the distributions are smoothed).

Method 3. A procedure is now outlined which throws light on the precise nature of the smoothing involved in deriving the distribution of v in this form.

We may write
$$\mathscr{E}(1-v) = \mathscr{E}\left(\frac{1}{1-r_1^2}\right) - \sum_{j=0}^{\infty} \mathscr{E}(r_1^{2j} r_2), \tag{6·5}$$
and it is easily established that
$$\mathscr{E}[(1-r_1^2)^{-1}] = \frac{n}{n-1},$$
giving the formal identity
$$\frac{n}{n-1} = \sum_{j=0}^{\infty} \mathscr{E}(r_1^{2j}) = 1 + \sum_{j=1}^{\infty} \frac{(2j-1)(2j-3)\dots 1}{(n+2)\dots(n+2j)}. \tag{6·6}$$
It will be shown in § 7 that
$$\mathscr{E}(r_1^{2j} r_2) = \frac{2j(2j-1)(2j-3)\dots 1}{(n+2)(n+4)\dots(n+2j+2)}.$$
(6·5) may therefore be written in the form
$$1 - \mathscr{E}(v) = \frac{n}{n-1} - \sum_{j=1}^{\infty} \frac{(2j-1)(2j-3)\dots 1}{(n+2)(n+4)\dots(n+2j)} + \sum_{j=1}^{\infty} \frac{(2j-1)(2j-3)\dots 1}{(n+4)(n+6)\dots(n+2j+2)},$$
which utilizing (6·6) yields
$$\mathscr{E}(v) = -\frac{1}{n+1}.$$
The implications of this procedure will be discussed in § 8.

7. The identification of the constructed distribution with the 'smoothed' form

(*a*) The moments of the constructed distribution may be obtained directly since
$$\mathscr{E}(r_1^{2s} r_2^t) = \int_{-1}^{+1}\int_{-1}^{+1} p(r_1)\, p(v)\, [v(1-r_1^2) + r_1^2]^t r_1^{2s} dr_1 dv,$$
and it may be shown that this expression is equivalent to
$$\begin{aligned}\delta[&\mathscr{E}(v^t)(n+1)(n+3)\dots(n+2t-1) + {}^tC_1 \mathscr{E}(v^{t-1})(n+1)\dots(n+2t-3)(2s+1) + \dots \\ &+ {}^tC_r \mathscr{E}(v^{t-r})(n+1)(n+3)\dots(n+2t-2r-1)(2s+1)(2s+3)\dots(2s+2r-1) + \dots \\ &+ (2s+1)(2s+3)\dots(2s+2t-1)],\end{aligned} \tag{7·1}$$
where
$$\delta = \frac{(2s-1)(2s-3)\dots 1}{(n+2)(n+4)\dots(n+2s+2t)}.$$
Thus we may derive $\mu'_{2s,t}$ for any particular value of t by substituting for $\mathscr{E}(v^t)$. In the case of $t = 1, 2$, (7·1) becomes
$$\mathscr{E}(r_1^{2s} r_2) = \frac{2s(2s-1)(2s-3)\dots 1}{(n+2)(n+4)\dots(n+2s+2)},$$
$$\mathscr{E}(r_1^{2s} r_2^2) = \frac{[(n+3)+(2s+1)^2](2s-1)(2s-3)\dots 1}{(n+2)(n+4)\dots(n+2s+4)}.$$

(*b*) We now proceed to evaluate the bivariate moments from the smoothed characteristic function
$$\phi = \exp\left[-\frac{n}{4\pi}\int_0^{2\pi} \log\{a + b\cos\alpha + c\cos 2\alpha\}\, d\alpha\right],$$

where $a = 1-\theta_0$, $b = -\theta_1$, $c = -\theta_2$. This is simplified by observing from the form of this function that

$$\left[\frac{\partial^{2s+t}\phi}{\partial\theta_1^{2s}\partial\theta_2^t}\right]_{\substack{\theta_1=0\\ \theta_2=0}} = \left[\frac{\partial^{2s}}{\partial\theta_1^{2s}}\left(\frac{\partial^t\phi}{\partial\theta_2^t}\right)_{\theta_2=0}\right]_{\theta_1=0}.$$

Our method is then to express $\partial^t\phi/\partial\theta_2^t$ as a function of θ_0 and θ_1 only and to expand the latter in the form of an infinite series in powers of $x = \theta_1/(1-\theta_0)$. The coefficient of x^{2s} in this series may then be integrated directly to give $\mathscr{E}(r_1^{2s}r_2^t)$.

Use could be made at this juncture of Dixon's form for ϕ, but it has been found easier to work with

$$\log\phi = -\frac{n}{4\pi}\int_0^{2\pi}\log\{a+b\cos\alpha+c\cos 2\alpha\}\,d\alpha,$$

and to perform the necessary operations on the integrand, the uniform convergence of the integral justifying this procedure.

It has been established that

$$\left[\frac{\partial\phi}{\partial\theta_2}\right]_{\theta_2=0} = \exp\left[-\frac{n}{4\pi}\int_0^{2\pi}\log(a+b\cos\alpha)\,d\alpha\right]\frac{n}{4\pi}\int_0^{2\pi}\frac{\cos 2\alpha}{a+b\cos\alpha}\,d\alpha,$$

which reduces to

$$\frac{i}{2}[\tfrac{1}{2}\{1+\sqrt{(1-x^2)}\}]^{-i-1}[(1-x^2)^{-\frac{1}{2}}-1], \quad \{i=\tfrac{1}{2}n\}.$$

These two quantities may be expanded in the form of infinite series in the following manner:

$$(1-x^2)^{-\frac{1}{2}}-1 = \sum_{j=1}^{\infty}\frac{(2j-1)(2j-3)\dots 1}{2^j j!}x^{2j}, \tag{7·2}$$

whilst for the first factor we utilize an expansion which is originally due to Laplace and which was used by Dixon in the univariate case, namely,

$$[\tfrac{1}{2}\{1+\sqrt{(1-x^2)}\}]^{-i-1} = \left[1+\frac{(i+1)x^2}{2^2}+\frac{(i+1)(i+4)}{2^2\,2!}x^4+\dots\right.$$
$$\left.+\frac{(i+1)(i+k+2)\dots(i+2k)}{k!\,2^{2k}}x^{2k}+\dots\right]. \tag{7·3}$$

The problem thus reduces to one of multiplying the two infinite series given by (7·2) and (7·3), and it may be shown that

$$\left[\frac{\partial\phi}{\partial\theta_2}\right]_{\theta_2=0} = \frac{1}{1-\theta_0}\left[\frac{i}{4}x^2+\sum_{j=2}^{\infty}\frac{i(i+j+2)(i+j+3)\dots(i+2j)x^{2j}}{(j-1)!\,2^{2j}}\right].$$

If the operations outlined previously are applied to the general term of this expansion, it may be verified that

$$\mathscr{E}(r_1^{2s}r_2) = \frac{2s(2s-1)(2s-3)\dots 1}{(n+2)(n+4)\dots(n+2s+2)}.$$

The procedure for $\partial^2\phi/\partial\theta_2^2$ is similar but more complicated, giving

$$\left[\frac{\partial^2\phi}{\partial\theta_2^2}\right]_{\theta_2=0} = \frac{1}{(1-\theta_0)^2}\left[\frac{i(i+6)}{2^3}x^2+\sum_{j=2}^{\infty}\frac{i(i+j+3)(i+j+4)\dots(i+2j+1)}{j!\,2^{2j+1}}[i+2(j^2+j+1)]x^{2j}\right],$$

which results in

$$\mathscr{E}(r_1^{2j}r_2^2) = \frac{[n+4(j^2+j+1)]\,(2j-1)\,(2j-3)\ldots 1}{(n+2)\,(n+4)\ldots(n+2j+4)}.$$

There is no point in proceeding any further, since these results are identical with those obtained from the constructed distribution.

It will be shown in the next section that the smoothing process always leads to correct moments for the serial correlations up to order n; it may thus be inferred that the same is true of the constructed distribution.

8. THE NATURE OF THE SMOOTHING OBTAINED

It is now important to consider the exact nature of the smoothing achieved in the joint distribution $p(r_1, r_2)$ and how this reflects itself in the distribution of v.

In essence, the smoothing process involves the replacement of the exact characteristic function

$$\phi = \exp\left[-\frac{1}{2}\sum_{k=1}^{n}\log\{a+b\cos\alpha+c\cos 2\alpha\}\right], \quad \left\{\alpha = \frac{2\pi k}{n}\right\},$$

by the smoothed function

$$\hat{\phi} = \exp\left[-\frac{n}{4\pi}\int_0^{2\pi}\log\{a+b\cos\alpha+c\cos 2\alpha\}\,d\alpha\right].$$

As was pointed out in § 5, the calculation of the exact moments reduces to the problem of evaluating the series

$$\sum_{k=1}^{n}\cos^s\frac{2\pi k}{n}\cos^t\frac{4\pi k}{n}.$$

It is now observed that in the case of the smoothed form, the moment problem is resolved by the solution of the integrals

$$\frac{n}{4\pi}\int_0^{2\pi}\cos^s\alpha\,\cos^t 2\alpha\,d\alpha,$$

and, in fact, the expressions given for the series in § 5 are only true for $\max(s,t) < n$, whilst they represent the value of the integral for all values of s and t. It thus follows that smoothed moments will agree with exact values provided this condition holds. The result of this procedure is to replace a function which is represented by different analytic forms in the various regions of the (r_1, r_2) plane by a single function defined over the region.

As far as the smoothing of the distribution of v is concerned, method 3 of § 6 illustrates the fact that the first moment of v depends on all moments of the form $\mathscr{E}(r_1^{2s}r_2)$ of the smoothed joint distribution. Since the latter are correct up to order n only, the first moment of v is not exact. However, the agreement will be very close, the moments differing from their exact values by quantities of order n^{-r}, where r itself is very large.

In an analogous manner it may be shown that in the case of the Leipnik distribution for r_1 in a Markoff scheme, namely,

$$p(r_1 \mid \alpha_1) = [B(\tfrac{1}{2}, \tfrac{1}{2}n+\tfrac{1}{2})]^{-1}(1-r_1^2)^{\frac{1}{2}(n-1)}\{1+\alpha_1^2-2\alpha_1 r_1\}^{-\frac{1}{2}n},$$

that

$$\mathscr{E}(r_1 \mid \alpha_1) = \frac{1}{(1+\alpha_1^2)^{\frac{1}{2}n}}\sum_{j=0}^{\infty}\frac{(\frac{1}{2}n+j-1)!}{(\frac{1}{2}n-1)!\,j!}\left(\frac{2\alpha_1}{1+\alpha_1}\right)^j\mu'_{j+1},$$

where

$$\mu'_{j+1} = \mathscr{E}(r_1^{j+1} \mid \alpha_1 = 0).$$

Since the latter are correct up to order n only, the same state of affairs obtains as in the case of v. Similar remarks may be made about the distribution of T^2_{20}, although the situation is not identical. It thus appears that whilst we may be able to smooth the joint distribution of serials, the moments of various functions of the latter may not agree with the exact values. Since the agreement is so close, the distributions of such functions may themselves be regarded from a practical point of view as smoothed distributions with high order moment agreement.

9. Modifications when the serials are corrected for the mean

In practice, it is only very rarely that we know the theoretical mean of a time series to be zero. It then becomes necessary to define a circular serial correlation coefficient corrected for the sample mean by

$$r_k = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(x_{i+k} - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

As in the univariate case, this procedure results in a complete loss of mathematical symmetry and the expressions for the moments are complicated.

It may be shown that the joint characteristic function of

$$\bar{p} = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2, \quad \bar{q}_1 = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})(x_{i+1}-\bar{x}), \quad \bar{q}_2 = \frac{1}{2\sigma^2}\sum(x_i-\bar{x})(x_{i+2}-\bar{x}),$$

is given by $$\bar{\phi}(\theta_0, \theta_1, \theta_2) = \prod_{k=1}^{n-1}\left\{1 - \theta_0 - \theta_1 \cos\frac{2\pi k}{n} - \theta_2 \cos\frac{4\pi k}{n}\right\}^{-\frac{1}{2}}.$$

The method described in § 5 is now immediately applicable, but it has been found convenient to modify the technique slightly. Using the same notation as in § 5, it follows that

$$\frac{1}{\bar{\phi}}\frac{\partial\bar{\phi}}{\partial\theta_1} = \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{\alpha}{A}\right),$$

so that

$$\frac{\partial^r\bar{\phi}}{\partial\theta_1^r} = \frac{\partial^{r-1}}{\partial\theta_1^{r-1}}\left[\frac{\bar{\phi}}{2}\sum_{k=1}^{n-1}\frac{\alpha}{A}\right],$$

and the latter may be evaluated using Leibnitz's theorem for the nth differential of the product of two functions. Since

$$\frac{\partial^{s-1}}{\partial\theta_1^{s-1}}\left[\sum_1^{n-1}\left(\frac{\alpha}{A}\right)\right] = (s-1)!\sum\left(\frac{\alpha}{A}\right)^s,$$

it may be shown that

$$\frac{\partial^r\bar{\phi}}{\partial\theta_1^r} = \frac{(r-1)!}{2}\sum_{j=0}^{r-1}\frac{1}{j!}\frac{\partial^j\bar{\phi}}{\partial\theta_1^j}\sum_{k=1}^{n-1}\left(\frac{\alpha}{A}\right)^{r-j}. \tag{9·1}$$

The values of the trigonometrical series may be obtained without further calculation, since

$$\sum_{k=1}^{n-1}\alpha^r\beta^s = \sum_{k=1}^{n}\alpha^r\beta^s - 1.$$

The moments may thus be built up systematically and the following results were obtained:

$$\mu'_{11} = -\frac{1}{n^2-1},$$

$$\mu'_{21} = \frac{n-3}{(n-1)(n+1)(n+3)},$$

$$\mu'_{31} = -\frac{3(3n+5)}{(n-1)(n+1)(n+3)(n+5)},$$

$$\mu'_{41} = \frac{3(3n^2+16n-35)}{(n-1)(n+1)(n+3)(n+5)(n+7)},$$

$$\mu'_{22} = \frac{n^2+4n-15}{(n-1)(n+1)(n+3)(n+5)}.$$

When it comes to the evaluation of the moments of $\bar{v}$, the following factors must be borne in mind:

(1) No smoothed form for $p(\bar{r}_1)$ exists, and it is thus necessary to resort to a two moment fitting of a Pearson type I curve.

(2) The assumption of independence of $\bar{r}_1$ and $\bar{v}$ is not tenable, since it leads to incorrect bivariate moments.

At this stage we can do no more than indicate the effect which the component $\bar{x}$ exerts on the distribution of $\bar{v}$.

It is possible to obtain approximations to these moments by using the expansion

$$1-v = \frac{1}{1-r_1^2} - \sum_{j=0}^{\infty} r_2 r_1^{2j}.$$

Since $\mathscr{E}(r_1^{2j} r_2)$ is of order n^{-j-1}, a good approximation to $\mathscr{E}(v)$ may be obtained by considering

$$1-\mathscr{E}(v) \doteqdot \mathscr{E}[(1-r_1^2)^{-1}] - \mathscr{E}(r_2) - \mathscr{E}(r_1^2 r_2)$$

and expanding the right-hand side in powers of $1/n$. For $\mathscr{E}(v^2)$ an analogous method is available by considering

$$(1-v)^2 = (1-r_1^2)^{-2} - 2r_2 + r_2^2 + (r_2^2 - 2r_2) \sum_{j=1}^{\infty} j r_1^j.$$

If this procedure is applied when the serials are not corrected for the mean, the values obtained for the first two moments of v coincide with the smoothed values derived previously.

When the serials are corrected for the mean, it was found that

$$\mathscr{E}(\bar{v}) \doteqdot -\frac{2}{n-1},$$

$$\mathscr{E}(\bar{v}^2) \doteqdot \frac{1}{n-2}.$$

Compared with the moments when no mean correction is made, it is seen that whereas the variance is slightly altered, the mean is approximately doubled. The experimental results in the next section tend to support this conclusion.

Dixon's results on the criterion $1-\bar{v}^2$ show that the even moments of $\bar{v}$ are obtained from those of v by replacing n by $n-1$. This effect has appeared in all distributions yet considered in the field of serial correlation, whilst correcting for the mean has a big effect on the odd

moments. Moreover, smoothed distributions are no longer available when a mean correction is made. It seems reasonable to suggest therefore that if the squares of the correlogram and the partial correlogram are used instead of the correlograms themselves, tests of significance will be available which are independent of the theoretical mean of the series.

10. Applications to artificial time series

The distributions considered have been those under the null hypothesis of randomness. In testing a Markoff scheme against a Yule scheme, for which v is the appropriate statistic, the null hypothesis is that the series follows a Markoff process.

By an application of a theorem due to Madow (1945), it is seen that the smoothed joint distribution of r_1 and r_2 in a Yule scheme is given by

$$p(r_1, r_2 \mid \alpha_1, \alpha_2) = p(r_1, r_2)\{1+\alpha_1^2+\alpha_2^2-2\alpha_1(1-\alpha_2)r_1-2\alpha_2 r_2\}^{-\frac{1}{2}n},$$

where $p(r_1, r_2)$ is the distribution when $\alpha_1 = 0 = \alpha_2$. Transforming gives

$$p(r_1, v \mid \alpha_1, \alpha_2) = p(r_1)\,p(v)\{1+\alpha_1^2+\alpha_2^2-2\alpha_1(1-\alpha_2)r_1-2\alpha_2 v(1-r_1^2)-2\alpha_2 r_1^2\}^{-\frac{1}{2}n},$$

where $$p(r_1) = p(r_1 \mid \alpha_1 = 0 = \alpha_2), \quad p(v) = p(v \mid \alpha_1 = 0 = \alpha_2).$$

In an exactly similar manner it follows that

$$\begin{aligned} p(r_1, v \mid \alpha_1) &= p(r_1)\,p(v)\{1+\alpha_1^2-2\alpha_1 r_1\}^{-\frac{1}{2}n} \\ &= p(r_1 \mid \alpha_1)\,\phi(v), \end{aligned}$$

which exhibits the fact that the distribution of v is the same in a Markoff scheme as in a random set-up. In the case where the serials are corrected for the mean, it seems that this will be very nearly true and the experimental results tend to confirm this.

Sampling experiments have been performed on the distribution of v in a Markoff scheme and a Yule scheme by Quenouille (1949*b*). From an analysis of several artificial series it was found that $\mathscr{E}(v)$ is always negative. This was termed 'bias' and artificial methods were used to remove this bias. This procedure is now seen to be unnecessary, since this effect may be explained by means of the distribution theory.

The following results, which are taken from the same paper relate to the mean and variance of v estimated from thirty subseries each containing twenty members, from a scheme which is approximately a Markoff process with $\alpha_1 = 0{\cdot}6$.

A circular definition was employed and the partial coefficients were calculated:

(1) Uncorrected for the mean.

(2) With a mean correction.

		(1)	(2)
$\mathscr{E}(v)$	Observed	−0·0489	−0·1443
	Expected	−0·0476	−0·1052
Var. (v)	Observed	0·0313	0·0411
	Expected	0·0451	0·0444

The agreement between observed and expected values is good except in the case of $\mathscr{E}(v)$ when the serials are corrected for the mean. It is hoped to provide further illustrations of the sampling distribution of v at a later stage when non-central distributions will be considered.

Finally, I am greatly indebted to Dr F. N. David and to Dr N. L. Johnson for their advice and encouragement during the preparation of this paper.

REFERENCES

DIXON, W. J. (1944). *Ann. Math. Statist.* **15**, 119.
JENKINS, G. M. (1954). *Biometrika*, **41**, 261.
MADOW, W. G. (1945). *Ann. Math. Statist.* **16**, 308.
MORAN, P. A. P. (1948). *Biometrika*, **35**, 255.
QUENOUILLE, M. H. (1947). *J.R. Statist. Soc.* **110**, 123.
QUENOUILLE, M. H. (1949*a*). *Ann. Math. Statist.* **19**, 561.
QUENOUILLE, M. H. (1949*b*). *J.R. Statist. Soc.* B, **11**, 68.
RAO, S. R. & SOM, K. S. (1951). *Sankhyā*, **11**, 239.
WHITTLE, P. (1952). *Biometrika*, **39**, 309.
YULE, G. U. (1927). *Phil. Trans.* A, **226**, 267.

SOME PROBLEMS OF OPTIMUM SAMPLING†

BY PAUL N. SOMERVILLE

Institute of Statistics, University of North Carolina and Virginia Polytechnic Institute

1. INTRODUCTION

An experimenter is asked which of $k+1$ populations has the largest mean and must decide how large a sample he should take to decide this question. Taking a large sample decreases the probability of an incorrect decision, but at the same time increases the cost of sampling. It seems obvious that the 'optimum' sample size should depend both on the cost of sampling and the amount of use to be made of the decision.

In this paper, loss functions are set up which take into consideration the amount of use to be made of the result, the cost of making a wrong decision and the cost of sampling. A theorem is proved which, under certain restrictions, enables us to find a maximum of the expected loss over all possible parameter values, the parameters being the means, variances, etc., of $k+1$ otherwise identically distributed populations. It is then possible to minimize this maximum loss with respect to the sample size. Results are given in the case where the parameters are means, and where the sample means are assumed to be normally distributed.

2. 'OPTIMUM' SAMPLE SIZE USING MINIMAX PROCEDURE

Suppose we have $k+1$ populations $\Pi_0, \Pi_1, \ldots, \Pi_k$, identically distributed except for unknown real-valued parameters $\theta_0 \geqslant \theta_1 \geqslant \ldots \geqslant \theta_k$. We wish to conduct some preliminary sampling procedure which will select the population with the largest‡ parameter, with the purpose of using it for a further 'sample' of size N. Let $W(\theta_i, \theta_0)$ be the loss incurred in selecting the population Π_i, where W is a real-valued function with continuous second derivative. Let

$$W(\theta_0, \theta_0) = 0, \quad W(\theta_i, \theta_0) \geqslant 0.$$

(There is no loss in generality in setting $W(\theta_0, \theta_0) = 0$, since we can always put

$$\overline{W}(\theta_i, \theta_0) = W(\theta_i, \theta_0) - W(\theta_0, \theta_0).)$$

Let a sample of size $n(k+1)$ be taken by an arbitrary procedure—one-stage, two-stage, etc. Let $C(n)$ be the cost of a sample of size $n(k+1)$, where $C(n)$ is a real-valued function. Let p_i be the probability that the population Π_i is chosen. Then, the expected loss is

$$L = \sum_{i=1}^{k} W(\theta_i, \theta_0) p_i + C(n). \tag{1}$$

The two theorems which follow are of a general nature and are intended to help in the determination of sets of parameter values which maximize L. Theorem 1 shows that given θ_0, a maximum of L may occur when $\theta_1, \theta_2, \ldots, \theta_k$ have a common value θ', which will in general depend on θ_0. Theorem 2 shows that if each $W(\theta_i, \theta_0)$ depends on a function $g_i = g(\theta_i, \theta_0)$ and each p_i can be expressed explicitly in terms of these g_i's, then a maximum of L may occur when $g_1, g_2, \ldots, g_k$ have a common value g', *independent* of θ_0,

† This research was done under a U.S. Air Force Contract, in part, monitored by the Office of Scientific Research.

‡ There is no essential difference in the problem of choosing the largest parameter and that of choosing the smallest parameter.

THEOREM 1. (*a*) For any given θ_0, if $\theta_i = \theta$ $(i = 1, 2, \ldots, k)$, then

$$\frac{\partial L}{\partial \theta_i} \equiv \frac{\partial L}{\partial \theta_j} \quad (i, j = 1, 2, \ldots, k).$$

(*b*) If there is a value θ' such that $\partial L/\partial \theta_j]_{\theta_i = \theta'} = 0$ $(j = 1, 2, \ldots, k)$, where $\theta_i = \theta'$ holds for all $i = 1, 2, \ldots, k$, then, if

$$\frac{\partial^2 L}{\partial \theta_j^2}\Big]_{\theta_i=\theta'} < 0 \quad \text{and} \quad \frac{-1}{k-1} < \left.\frac{\dfrac{\partial^2 L}{\partial \theta_j \partial \theta_m}}{\dfrac{\partial^2 L}{\partial \theta_j^2}}\right]_{\theta_i=\theta'} < 1,$$

for all j and m, a maximum of L exists at this point.

Proof of (*a*). It follows from equation (1) that

$$\frac{\partial L}{\partial \theta_j} = W'(\theta_j, \theta_0)\, p_j + \sum_{i=1}^{k} W(\theta_i, \theta_0) \frac{\partial p_i}{\partial \theta_j} \quad (j = 1, 2, \ldots, k). \tag{2}$$

Since $\sum_{i=0}^{k} p_i = 1$, we have $\quad \sum_{i=1}^{k} \frac{\partial p_i}{\partial \theta_j} = -\frac{\partial p_0}{\partial \theta_j} \quad (j = 1, 2, \ldots, k),$

and therefore

$$\frac{\partial L}{\partial \theta_j}\Big]_{\theta_i=\theta} = W'(\theta, \theta_0)\, p_j\Big]_{\theta_i=\theta} - W(\theta, \theta_0) \frac{\partial p_0}{\partial \theta_j}\Big]_{\theta_i=\theta} \quad (i, j = 1, 2, \ldots, k).$$

But, if $\theta_i = \theta_j$ $(i, j = 1, 2, \ldots, k)$, we have by symmetry,

$$p_i = p_j = p\,(\text{say}) = (1 - p_0)/k.$$

Similarly $\quad \frac{\partial p_0}{\partial \theta_j}\Big]_{\theta_i=\theta} = \frac{\partial p_0}{\partial \theta_m}\Big]_{\theta_i=\theta} \quad (i, j = 1, 2, \ldots, k).$

Therefore $\quad \frac{\partial L}{\partial \theta_j}\Big]_{\theta_i=\theta} \equiv \frac{\partial L}{\partial \theta_m}\Big]_{\theta_i=\theta} \quad (i, j, m = 1, 2, \ldots, k).$

Proof of (*b*). Proceeding from equation (2), we find

$$\frac{\partial^2 L}{\partial \theta_j \partial \theta_m} = W''(\theta_j, \theta_0)\, p_j \delta_{jm} + W'(\theta_j, \theta_0) \frac{\partial p_j}{\partial \theta_m} + W'(\theta_m, \theta_0) \frac{\partial p_m}{\partial \theta_j} + \sum_{i=1}^{k} W(\theta_i, \theta_0) \frac{\partial^2 p_i}{\partial \theta_j \partial \theta_m},$$

where $\quad \delta_{ij} = 0, \quad i \neq j \quad \text{and} \quad \delta_{ii} = 1.$

Putting $\theta_i = \theta'$ $(i = 1, 2, \ldots, k)$, and using considerations of symmetry, we have

$$\frac{\partial^2 L}{\partial \theta_j \partial \theta_m}\Big]_{\theta_i=\theta'} = W''(\theta', \theta_0)\, p_j \delta_{jm} + 2W'(\theta', \theta_0) \frac{\partial p_j}{\partial \theta_m}\Big]_{\theta_i=\theta'} - W(\theta_i, \theta_0) \frac{\partial^2 p_0}{\partial \theta_j \partial \theta_m}\Big]_{\theta_i=\theta'}.$$

We shall have a maximum if and only if the matrix of second partials of L with respect to θ_j, θ_m, evaluated at $\theta_1 = \theta_2 = \ldots = \theta_k = \theta'$, is negative definite.

The matrix of the second partials may be written

$$\frac{\partial^2 L}{\partial \theta_j^2}\Big]_{\theta_i=\theta'} \times \begin{pmatrix} 1 & r & \ldots & r \\ r & 1 & \ldots & r \\ \vdots & & \ddots & \\ r & r & \ldots & 1 \end{pmatrix} = \frac{\partial^2 L}{\partial \theta_j^2}\Big]_{\theta_i=\theta'} (A_k), \quad \text{say,}$$

where $\quad r = \left.\dfrac{\dfrac{\partial^2 L}{\partial \theta_j \partial \theta_m}}{\dfrac{\partial^2 L}{\partial \theta_j^2}}\right]_{\theta_i=\theta'}.$

Now the determinant of A_k can easily be shown to have the value $[1+(k-1)r][1-r]^{k-1}$, from which it follows that (A_k) is definite (and positive) if, and only if, $\frac{-1}{k-1} < r < 1$. Therefore

$$\left.\frac{\partial^2 L}{\partial \theta_j^2}\right]_{\theta_i=\theta'} (A_k)$$

is negative definite if and only if

$$\left.\frac{\partial^2 L}{\partial \theta_j^2}\right]_{\theta_i=\theta'} < 0 \quad \text{and} \quad \frac{-1}{k-1} < \left.\frac{\dfrac{\partial^2 L}{\partial \theta_j \partial \theta_m}}{\dfrac{\partial^2 L}{\partial \theta_j^2}}\right]_{\theta_i=\theta'} < 1.$$

Let there exist a function $g = g(\theta_i, \theta_0) = g_i$ (say) such that

$$p_i = p_i(\theta_0, \theta_1, \ldots, \theta_k) = p_i(g_1, g_2, \ldots, g_k).$$

Then, if $W(\theta_i, \theta_0) = W(g_i)$, we have

THEOREM 2. (*a*) If $g_i = g$ $(i = 1, 2, \ldots, k)$, then

$$\frac{\partial L}{\partial g_i} \equiv \frac{\partial L}{\partial g_j} \quad (i, j = 1, 2, \ldots, k).$$

(*b*) If there is a value g' such that

$$\left.\frac{\partial L}{\partial g_j}\right]_{g_i=g'} = 0 \quad (j = 1, 2, \ldots, k),$$

then if

$$\left.\frac{\partial^2 L}{\partial g_j^2}\right]_{g_i=g'} < 0 \quad \text{and} \quad \frac{-1}{k-1} < \left.\frac{\dfrac{\partial^2 L}{\partial g_j \partial g_m}}{\dfrac{\partial^2 L}{\partial g_j^2}}\right]_{g_i=g'} < 1,$$

a maximum value of L exists at this point (i.e. independently of θ_0).

Proof. The proof is the same as the proof of Theorem 1 with θ_i replaced by g_i, and $W(\theta_i, \theta_0)$ replaced by $W(g_i)$.

Assuming that the g_i's exist, we have

$$L = \sum_{i=1}^{k} W(g_i) p_i + C(n),$$

$$\frac{\partial L}{\partial g_j} = W'(g_j) p_j + \sum_{i=1}^{k} W(g_i) \frac{\partial p_i}{\partial g_j} \quad (j = 1, 2, \ldots, k),$$

$$\left.\frac{\partial^2 L}{\partial g_j \partial g_m}\right]_{g_i=g'} = W''(g') p \delta_{jm} + 2W'(g') \left.\frac{\partial p_j}{\partial g_m}\right]_{g_i=g'} - W(g') \left.\frac{\partial^2 p_\sigma}{\partial g_j \partial g_m}\right]_{g_i=g'} \quad (i, j, m = 1, 2, \ldots, k),$$

where, as before, $\delta_{ij} = 0$ $(i \neq j)$ and $\delta_{ii} = 1$.

Theorem 2 enables us to find a maximum of the expected loss over all possible parameter values. We have not proved that this maximum is the only maximum, nor even that it is the largest maximum. As a matter of fact, it is possible to imagine functions W which could yield more than one maximum. However, in the case where the unknown parameters are means or variances of normal distributions, and the W's are function of the type $c_1(\theta_0 - \theta_i)^{c_2}$, $c_1 e^{c_2(\theta_0 - \theta_i)}$ or other seemingly 'reasonable' loss functions, calculations lead one to believe that the maximum will in fact be unique.

The above maximum loss can then be minimized with respect to the preliminary sample size.

3. ONE-STAGE SAMPLING FOR MEANS

Suppose, as in § 2, that we have $k+1$ populations $\Pi_0, \Pi_1, \ldots, \Pi_k$, identically distributed except for unknown means $\theta_0 \geqslant \theta_1 \geqslant \ldots \geqslant \theta_k$, and with the second moment of Π_i about θ_i equal to σ^2. It is desired to use the population with the largest mean for N occasions. A preliminary sample of n is taken from each population, with the object of deciding which population is Π_0. The N individuals are then chosen from the population yielding the largest preliminary sample mean.† As before, let $C(n)$ be the cost of taking the preliminary sample, and $W(\theta_i, \theta_0)$ be the loss incurred in choosing population Π_i. Let $\bar{x}_i$ be the mean of the sample from Π_i, and let p_i be the probability that the sample from Π_i will have the largest mean. The total expected loss is thus

$$L = \sum_{i=1}^{k} p_i W(\theta_i, \theta_0) + C(n),$$

where $$p_i = \Pr\{\bar{x}_0 < \bar{x}_i, \ldots, \bar{x}_{i-1} < \bar{x}_i, \bar{x}_{i+1} < \bar{x}_i, \ldots, \bar{x}_k < \bar{x}_i\} \quad (i = 1, 2, \ldots, k).$$

Let $$g_i = \frac{\theta_0 - \theta_i}{\sigma} (\tfrac{1}{2}n)^{\frac{1}{2}},$$

$f(x_1, \ldots, x_k)$ be the multivariate normal frequency function with variances equal to 1 and covariances equal to $\frac{1}{2}$, and let $F(a_1, \ldots, a_k)$ be the integral of $f(x_1, \ldots, x_k)$ over the region $-\infty < x_i < a_i$ $(i = 1, \ldots, k)$. Then, if we assume $\bar{x}_i$ is normally distributed (which will usually be a reasonable assumption even when the populations Π_i are non-normal), we can show that

$$p_0 = F(g_1, \ldots, g_k),$$
$$p_i = F(-g_i + g_1, \ldots, -g_i + g_{i-1}, -g_i, -g_i + g_{i+1}, \ldots, -g_i + g_k) \quad (i = 1, 2, \ldots, k).$$

If our object is to make as large as possible the expected value of the N individuals, then a reasonable loss function would seem to be

$$W(\theta_i, \theta_0) = N(\theta_0 - \theta_i) = N\sigma g_i \left(\frac{2}{n}\right)^{\frac{1}{2}} = W(g_i).$$

Using this, and putting $g_i = g$ (see Theorem 2) $(i = 1, 2, \ldots, k)$,

$$L = kN\sigma g p_i \left(\frac{2}{n}\right)^{\frac{1}{2}} + C(n)$$
$$= N\sigma \left(\frac{2}{n}\right)^{\frac{1}{2}} g(1 - p_0) + C(n).$$

Theoretically, the value at which the maximum occurs can be found by solving

$$\left.\frac{\partial L}{\partial g_i}\right]_{g_i = g} = 0.$$

However, practically, it seems easier to evaluate L for various values of g, and using methods of finite differences, obtain the maximum value of L with respect to g in that manner.

Using the generalized Mehler Series of Kibble (1945), $F(g_1, \ldots, g_k)$ has been calculated for $k = 3, 4, 5$. Results are shown in Table 3·1 along with results for $k = 1, 2$, which can be found in Karl Pearson's (1931) *Tables for Statisticians and Biometricians*, vol. 1, Table II and vol. 2, Table VIII respectively.

† This procedure is justified by Bahadur (1950).

Using Table 3·1 and Salzer (1948) *Tables of Coefficients for Obtaining the First Derivative Without Differences* and the Laplace-Everett central-difference formula, the maxima of $g(1-p_0)$, and the maximizing values $g' = g_1 = \ldots = \ldots g_k$, were calculated and are shown in Table 3·2.

Table 3·1. *Multivariate normal integral*

x	$F(x)$	$F(x,x)$	$F(x,x,x)$	$F(x,x,x,x)$	$F(x,x,x,x,x)$
0·0	0·50000	0·33333	0·2500	0·2000	0·167
0·1	0·53983	0·37408	0·2875	0·2341	0·198
0·2	0·57926	0·41623	0·3275	0·2714	0·232
0·3	0·61791	0·45931	0·3697	0·3117	0·271
0·4	0·65542	0·50282	0·4137	0·3544	0·312
0·5	0·69146	0·54625	0·4586	0·3990	0·356
0·6	0·72575	0·58906	0·5040	0·4449	0·401
0·7	0·75804	0·63079	0·5491	0·4912	0·447
0·8	0·78814	0·67098	0·5935	0·5373	0·494
0·9	0·81594	0·70922	0·6364	0·5827	0·541
1·0	0·84134	0·74520	0·6777	0·6269	0·587
1·1	0·86433	0·77866	0·7169	0·6695	0·632
1·2	0·88493	0·80941	0·7536	0·7100	0·675
1·3	0·90320	0·83734	0·7876	0·7481	0·716
1·4	0·91924	0·86243	0·8186	0·7834	0·754
1·5	0·93319	0·88471	0·8466	0·8155	0·789
1·6	0·94520	0·90427	0·8715	0·8443	0·821
1·7	0·95543	0·92124	0·8934	0·8699	0·850
1·8	0·96407	0·93581	0·9124	0·8923	0·875
1·9	0·97128	0·94817	0·9287	0·9119	0·897
2·0	0·97725	0·95855	0·9425	0·9286	0·916
2·5	0·99379	0·98825	0·9832	0·9785	0·974
3·0	0·99865	0·99736	0·9961	0·9951	0·994

Table 3·2. *Maximum of* $g(1-p_0)$

k	$\max_g g(1-p_0)$	g'
1	0·169971	0·7518
2	0·2634	0·8238
3	0·3274	0·878
4	0·376	0·92
5	0·414	0·95

Note that we have not yet demonstrated that we have a maximum of L and not merely a stationary point. By Theorem 2 we will have a true maximum if

$$\left.\frac{\partial^2 L}{\partial g_j^2}\right]_{g_i=g'} = G_0(g') < 0,$$

and

$$\frac{-1}{k-1} < \left.\frac{\dfrac{\partial^2 L}{\partial g_j \partial g_m}}{\dfrac{\partial^2 L}{\partial g_j^2}}\right]_{g_i=g'} = G_1(g') < 1.$$

By calculation, we find the following values for $G_0(g')$ and $G_1(g')$ when $g(1-p_0)$ is maximized. Thus the conditions of Theorem 2 are satisfied, and we have actually demonstrated the existence of the maxima.

Table 3·3

k	$G_0(g')$	$G_1(g')$
1	−0·4315	—
2	−0·343	−0·115
3	−0·291	−0·091
4	−0·252	−0·07
5	−0·22	−0·06

Let M_k be the maximum of $g(1-p_0)$. Then

$$\max_g L = N\sigma M_k\left(\frac{2}{n}\right)^{\frac{1}{2}} + C(n).$$

Let the cost of taking the preliminary sample be a linear function of the sample size, i.e. $C(n) = c_1 n + c_0$. Minimizing this maximum with respect to n, we have

$$\frac{\partial L}{\partial n} = -2^{-\frac{1}{2}} N\sigma M_k n^{-\frac{1}{2}} + c_1$$

$$= 0 \quad \text{if} \quad n^{-\frac{1}{2}} = \frac{c_1\sqrt{2}}{\sigma N M_k},$$

i.e. if

$$n = \left(\frac{\sigma M_k}{c_1\sqrt{2}}\right)^{\frac{1}{2}} N^{\frac{1}{2}}.$$

This means that the optimum n is proportional to $N^{\frac{1}{2}}$, a result noted by Robbins (see also Bross, 1950) for the special case of $k = 1$, that is, for two variates.

The expected loss at this value of n is

$$L = \sqrt{2}\left(\frac{\sigma M_k}{c_1\sqrt{2}}\right)^{-\frac{1}{2}} N^{-\frac{1}{2}}\sigma N M_k + c_1\left(\frac{\sigma M_k}{c_1\sqrt{2}}\right)^{\frac{1}{2}} N^{\frac{1}{2}} + c_0$$

$$= 2^{-\frac{1}{4}}\sigma^{\frac{1}{2}} M_k^{\frac{1}{2}} c_1^{\frac{1}{2}} N^{\frac{1}{2}}[2+1] + c_0 = 2{\cdot}3811 M_k^{\frac{1}{2}}\sigma^{\frac{1}{2}} c_1^{\frac{1}{2}} N^{\frac{1}{2}} + c_0.$$

It is worth noting that one-third of the maximum expected loss (neglecting c_0) is due to sampling costs and two-thirds due to loss from choosing the wrong population. This maximum occurs where

$$\theta_0 - \theta_i = \theta' = \left(\frac{2}{n}\right)^{\frac{1}{2}}\sigma g' = 2^{\frac{1}{4}} c_1^{\frac{1}{2}}\sigma^{\frac{1}{2}} M_k^{-\frac{1}{2}} N^{-\frac{1}{2}} g'.$$

We may then construct Table 3·4.

As shown in the table, the maximum value of $g(1-p_0)$ occurs at values of θ_i (such that $\sigma^{-1}(\theta_0 - \theta_i) \doteq 2(c_1/N\sigma)^{\frac{1}{2}}$) which are not usually improbably large. In other words, it is quite possible for the maximum possible loss to occur, and thus our minimization of the maximum appears reasonable.

It will be noted that the optimum sample size as expressed is an explicit function of the use to be made of the sampling result, and of the sampling cost per unit. Further, the loss per unit is proportional to $N^{-\frac{1}{2}}$. Thus, as N increases, the loss per unit decreases. This is in contrast to the usual manner of selecting sample size where a minimum δ which it is desired

to detect, and a probability β of detecting this δ, are set up. In the latter case, the selected sample size is a function only of the β and δ, and the loss per unit is fixed.

It should be stated that we have not proved that the maximum we have obtained is a unique maximum, but only that it is a local maximum. In case the populations are normal, and using our W, it is easily proved that the maximum is unique for $k = 1$, and for $k = 2, 3, 4, 5$ sufficient calculations have been made that it seems the maximum is unique.

Table 3·4. *Values of M_k, n, $L - c_0$, $\theta_0 - \theta_i$ for minimizing the maximum expected loss*

k	M_k	Optimum n (multiply by $\sigma^{\frac{4}{3}} c_1^{-\frac{1}{3}} N^{\frac{1}{3}}$)	Maximum expected loss $- c_0$ (multiply by $\sigma^{\frac{4}{3}} c_1^{\frac{1}{3}} N^{\frac{1}{3}}$)	Value of $\theta_0 - \theta_i$ at which maximum occurs (multiply by $\sigma^{\frac{1}{3}} c_1^{\frac{1}{3}} N^{-\frac{1}{3}}$)
1	0·16997	0·24354	0·73062	2·154
2	0·26344	0·32617	0·97851	2·05
3	0·3274	0·3767	1·131	2·01
4	0·376	0·412	1·24	2·0
5	0·414	0·43	1·3	2·0

It is possible to use different W's such as

$$c_2(\theta_0 - \theta_i)^r, \quad c_2 e^{c_3(\theta_0 - \theta_i)^r}, \quad \text{etc.}$$

From calculations, it would seem that for normal populations the maxima obtained are unique in these cases also.

Suppose we wish to choose the population having the largest mean but are concerned only with detecting a difference at least as great as δ.

Suppose also that we consider any differences greater than or equal to δ, equally undesirable. We may put

$$W(\theta_i, \theta_0) = 1 \quad \text{if} \quad \theta_0 - \theta_i \geqslant \delta$$
$$= 0 \quad \text{if} \quad \theta_0 - \theta_i < \delta \quad (i = 1, 2, \ldots, k).$$

Bechhofer, Dunnett & Sobel (1954) consider this problem, finding the smallest sample size n which will detect the largest mean with probability at least $1 - \beta$.

Using the above function for W, and assuming the sample mean from Π_i is normally distributed, it is not difficult to show that there exists a value $\theta_1 = \theta_2 = \ldots = \theta_k = \theta (= \theta_0 - \delta)$ for which L is a unique maximum. The expected loss from choosing the wrong population is

$$L^* = \sum_{i=1}^{k} W(\theta_i, \theta_0) p_i.$$

Suppose $\theta_0 - \theta_i < \delta \quad (i = 1, 2, \ldots, m),$

$$\theta_0 - \theta_i \geqslant \delta \quad (i = m+1, m+2, \ldots, k).$$

Therefore $$L^* = p_{m+1} + p_{m+2} + \ldots + p_k.$$

Differentiating L^* with respect to g_i, it is easy to show that L^* is an increasing function of g_i for $i = 1, 2, \ldots, m$, and a decreasing function of g_i for $i = m+1, m+2, \ldots, k$. Thus, L^* is a maximum (uniquely) where

$$\theta_0 - \theta_i = \delta \quad (i = 1, 2, \ldots, k).$$

It is interesting to examine the consequences of not sampling at all. Suppose that we are willing to state that $\theta_k \geqslant \theta_s$, $\theta_0 \leqslant \theta_g$, where θ_s, θ_g represent the smallest and the largest means we can imagine the population to possess. Then the maximum expected loss by not sampling, i.e. choosing a population at random, is

$$N(\theta_g - \theta_s)\frac{k}{k-1}.$$

Now the expected loss at our minimax n is

$$2{\cdot}3811 M_k^{\frac{1}{2}} \sigma^{\frac{1}{2}} c_1^{\frac{1}{2}} N^{\frac{1}{2}} + c_0.$$

If the latter expression is larger than the former, then it seems reasonable that we should not sample at all. This might occur when the cost of sampling is high, either because of the initial cost c_0, or the additional cost per unit sampled c_1, or where N is small. An example might be a manufacturer who is to produce an expensive product which will have a limited use. To sample by trying out two or more methods of production, or variations of the same product, might be extremely expensive compared with the amount gained by using the improved product.

It is also of interest to examine the consequences of not using the minimax n, but instead rn, where $r > 0$. Our maximum expected loss becomes

$$L = 2^{-\frac{1}{2}} \sigma^{\frac{1}{2}} M_k^{\frac{1}{2}} c_1^{\frac{1}{2}} N^{\frac{1}{2}} [2r^{-\frac{1}{2}} + r] + c_0.$$

Neglecting c_0, L is increased in the ratio $\frac{1}{3}(2r^{-\frac{1}{2}} + r)$. For $0{\cdot}5 < r < 1{\cdot}8$, the maximum L is then increased by less than 10% of its value at $r = 1$. Thus, if the experimenter uses as little as one half the minimax n, he will increase his maximum L by only 10%. However, instead of one-third of L being due to the cost of sampling, and the remainder due to the loss in choosing the wrong variety, now 0·18 of the loss is due to the cost of sampling, and 0·82 is due to the loss in choosing the wrong variety (again neglecting c_0). If r becomes smaller than 0·5, the maximum L increases very rapidly.

4. TWO-STAGE SAMPLING FOR MEANS

Suppose we have three normally distributed populations Π_0, Π_1, Π_2 with variances σ^2 and unknown means $\theta_0 \geqslant \theta_1 \geqslant \theta_2$ respectively. It is desired, as in § 3, to choose N individuals from Π_0. We are allowed a preliminary sample of $3n$ to determine which population is Π_0. Using the notation of the preceding sections, we write the expected loss as

$$L = \sum_{i=1}^{2} W(\theta_i, \theta_0)\, p_i + C(n),$$

where p_i is the probability that we choose the N individuals from Π_i, and $W(\theta_i, \theta_0)$ is the loss involved in choosing them from Π_i.

Let us suppose we take a two-stage sample as follows:

Take a sample of size n_1 from each of the three populations. Discard the population having the smallest sample mean in the first stage, and take a further sample of n_2 from each of the two remaining populations, where $3n_1 + 2n_2 = 3n$. Choose the N individuals from the population having the largest cumulative sample mean.

Let p_i be the probability that the N individuals are selected from Π_i. Then it can be shown that

$$p_0 = F(\alpha g_2, g_1 - g_2, g_1; A) + F(\alpha g_1, g_2 - g_1, g_2; A),$$
$$p_1 = F(\alpha[g_2 - g_1], -g_2, -g_1; A) + F(-\alpha g_1, g_2, g_2 - g_1; A),$$
$$p_2 = F(\alpha[g_1 - g_2], -g_1, -g_2; A) + F(-\alpha g_2, g_1, g_1 - g_2; A),$$

where $$g_i = \frac{\theta_0 - \theta_i}{\sigma}(\tfrac{1}{2}n_1)^{\frac{1}{2}}, \quad \alpha = \left(\frac{n_1+n_2}{n_1}\right)^{\frac{1}{2}},$$

and F represents the integral of a multivariate normal frequency function with variance-covariance matrix A given by

$$A = \begin{pmatrix} 1 & \frac{-1}{2\alpha} & \frac{1}{2\alpha} \\ \frac{-1}{2\alpha} & 1 & \frac{1}{2} \\ \frac{1}{2\alpha} & \frac{1}{2} & 1 \end{pmatrix}.$$

As in § 3, we put $$W(\theta_i, \theta_0) = N(\theta_0 - \theta_i) = N\sigma g_i\left(\frac{2}{n_1}\right)^{\frac{1}{2}}.$$

Thus, by Theorem 2, we know that a maximum of L occurs for some $g' = g_1 = g_2$ provided the conditions on the second derivatives are satisfied.

Putting $g_1 = g_2 = g$, $$L = 2N\sigma g(1-p_0)\left(\frac{2}{n_1}\right)^{\frac{1}{2}} + C(n).$$

Further, since $3n = 3n_1 + 2n_2$ and $(n_1+n_2)/n_1 = \alpha^2$, we have

$$n_1 = \frac{3n}{2\alpha^2+1}, \quad \frac{n_1}{n_2} = \frac{1}{\alpha^2-1}.$$

Thus $$L = N\left(\frac{2\alpha^2+1}{3}\right)^{\frac{1}{2}}\left(\frac{2}{n}\right)^{\frac{1}{2}}\sigma g(1-p_0) + C(n).$$

It will be noted that L here differs from the L in the one-stage procedure only in the factor $(2\alpha^2+1)^{\frac{1}{2}}/\sqrt{3}$, which is equal to one in the limiting case $\alpha = 1$, where the two-stage sample is actually a one-stage sample.

Again using the generalized Mehler series of Kibble (1945), we may evaluate p_0, and thus calculate $$M = \max_g [\tfrac{1}{3}(2\alpha^2+1)]^{\frac{1}{2}} g(1-p_0),$$

obtaining the results shown in Tables 4·1 and 4·2.

Graphing the results in Table 4·2, it is seen that the two-stage sampling procedure has a smaller maximum expected loss than the one-stage procedure using the same total sample size, provided $1 \leqslant \alpha \leqslant 1{\cdot}92$. The smallest maximum loss occurs where $\alpha = 1{\cdot}35$.

Table 4·1. *Values of* $p_0 = 2F(\alpha g, 0, g; A)$

g \ α	1	1·2	1·3	1·4	1·5	1·8	1·9
0·3	—	—	—	—	—	—	—
0·4	—	—	—	—	0·5677	0·6060	0·6168
0·5	—	—	0·5992	0·6141	0·6299	0·6702	0·6822
0·6	—	0·6318	0·6509	0·6686	0·6854	0·7292	0·7418
0·7	0·6308	0·6789	0·6999	0·7207	0·7370	0·7815	0·7940
0·8	0·6710	0·7232	0·7455	0·7687	—	—	—
0·9	0·7092	0·7643	—	—	—	—	—
1·0	0·7452	—	—	—	—	—	—

Table 4·2. $M = \max_g [\frac{1}{3}(2\alpha^2+1)]^{\frac{1}{2}} g(1-p_0)$

α	1	1·2	1·3	1·4	1·5	1·8	1·9
M	0·2634	0·2558	0·2547	0·2547	0·2556	0·2607	0·2632
Max at $g =$	0·834	0·710	0·657	0·614	0·594	0·521	0·503

Since $n_1/n_2 = (\alpha^2-1)^{-1}$, we have the result that the two-stage sampling procedure has a smaller maximum expected loss provided $0·37 \leqslant n_1/n_2 \leqslant \infty$ and the maximum is smallest where $n_1/n_2 = 1·22$.

Suppose we equate the maximum expected losses for the one- and two-stage procedures by adjusting the total sample size for the two-stage procedure.

Let n^* be the total sample size for the two-stage procedure, and put $n_1/n_2 = 1·22$. Equating the maximum expected losses, we obtain

$$0·2634/n^{\frac{1}{2}} = 0·2546/n^{*\frac{1}{2}} \quad \text{or} \quad n^*/n = 0·934,$$

i.e. we effect a saving in sample size of 6·6 %.

We have not yet shown that we have a maximum of L and not merely an inflexion point. However, an extremely lengthy piece of algebra confirms that condition (b) of Theorem 2 is in fact satisfied.

This research was suggested by Dr H. E. Robbins. The author is indebted to Dr N. L. Johnson for his encouragement and valuable suggestions and criticism during the course of the work.

REFERENCES

BAHADUR, R. R. (1950). *Ann. Math. Statist.* **21**, 362.

BECHHOFER, R. E., DUNNETT, C. W. & SOBEL, M. (1954). *Biometrika*, **41**, 170, and other unpublished work.

BROSS, IRWIN (1950). *J. Amer. Statist. Ass.* **45**, 50.

KIBBLE, W. F. (1945). *Proc. Camb. Phil. Soc.* **41**, 12.

PEARSON, KARL (1931). *Tables for Statisticians and Biometricians*. London: Biometrika Trust.

SALZER, H. E. (1948). *Tables of Coefficients for Obtaining the First Derivative Without Differences*, U.S. Government Printing Office, Washington, National Bureau of Standards Applied Mathematics, Series 2.

A TEST FOR MARKOFF CHAINS

By PAUL G. HOEL

University of California, Los Angeles

1. Introduction

By proving the asymptotic normality of certain frequency counts in Markoff chains, Bartlett (1951) was able to construct a likelihood ratio test for the goodness of fit of such chains. His test is designed to test whether a sequence of observations is at most r-dependent. In it he assumes that the transition probabilities are known, or at least depend upon a limited number of parameters which can be estimated. If the transition probabilities are completely unknown, a different test is needed. It is the purpose of this paper to present such a test. The derivation depends heavily upon Bartlett's results and methods, and is essentially a modification and amplification of some of his methods.

2. Derivation

Assume that a sequence of observations $x_1, x_2, \ldots, x_n$ can be treated as the sample values of a Markoff chain of order r or less, and consider the problem of testing the hypothesis H_0 that a chain of order $r-1$ will suffice. More precisely, if $p_{ij\ldots kl}$ denotes the transition probability for an r-chain, the hypothesis to be tested is

$$H_0: \quad p_{ij\ldots kl} = p_{j\ldots kl} \quad (i = 1, 2, \ldots, s).$$

By a chain of order r is meant a chain in which the dependence extends over $r+1$ consecutive variables only. For such a chain there will therefore be $r+1$ subscripts in the transition probability $p_{ij\ldots kl}$.

By choosing r sufficiently large to include any reasonable model, it is possible to test successively lower orders until an order is reached below which it is unreasonable to go. Such a test essentially determines the order of the dependence. It is not capable of testing the adequacy of the Markoff chain assumption itself with respect to such things as the homogeneity of the sequence, etc. This criticism also applies to Bartlett's test.

The test that will be constructed is an asymptotic version of the likelihood ratio test for composite hypotheses. Now, following Bartlett, the likelihood function is, except for neglected beginning terms, given by

$$L = \prod_{i,\ldots,l} p_{ij\ldots kl}^{n_{ij\ldots kl}}, \tag{1}$$

where the indices range from 1 to s, corresponding to s possible states, and where $n_{ij\ldots kl}$ denotes the observed frequency of the r-chain state $ij\ldots kl$. The maximum-likelihood estimate of $p_{ij\ldots kl}$ is

$$\hat{p}_{ij\ldots kl} = \frac{n_{ij\ldots kl}}{n_{ij\ldots k}}, \tag{2}$$

where $n_{ij\ldots k} = \sum_{l=1}^{s} n_{ij\ldots kl}$. Using primes to denote parameter values under H_0, it follows from (2) that

$$\hat{p}'_{ij\ldots kl} = \hat{p}_{j\ldots kl} = \frac{n_{j\ldots kl}}{n_{j\ldots k}}. \tag{3}$$

The likelihood ratio for testing H_0 then assumes the form

$$\lambda = \frac{L_0(\text{max.})}{L(\text{max.})} = \frac{L(\hat{p}'_{ij\dots kl})}{L(\hat{p}_{ij\dots kl})}. \tag{4}$$

Bartlett (1951) has shown that the $n_{ij\dots kl}$ are asymptotically normally distributed under mild regularity conditions; hence, assuming those conditions,

$$L \sim c\,|\,A\,|^{\frac{1}{2}} e^{-\frac{1}{2}[n-\mu]A(n-\mu)}. \tag{5}$$

Here $[n-\mu]$ denotes the row vector and $(n-\mu)$ the column vector of the linearly independent variables $n_{ij\dots kl}-\mu_{ij\dots kl}$, where $\mu_{ij\dots kl}$ is the expected value of $n_{ij\dots kl}$. This notation is used in place of the customary prime notation for transposition because primes are being used to denote values under H_0. The positive definite matrix A and the vector μ are functions of the parameters $p_{ij\dots kl}$. Applying (5) to (4), λ will assume the asymptotic form

$$\lambda \sim \frac{|\,\hat{A}'\,|^{\frac{1}{2}} e^{-\frac{1}{2}[n-\hat{\mu}']\hat{A}'(n-\hat{\mu}')}}{|\,\hat{A}\,|^{\frac{1}{2}} e^{-\frac{1}{2}[n-\hat{\mu}]\hat{A}(n-\hat{\mu})}},$$

where $\hat{\mu}$, $\hat{A}$ and $\hat{\mu}'$, $\hat{A}'$ indicate that the parameters $p_{ij\dots kl}$ have been replaced by their maximum likelihood estimates $\hat{p}_{ij\dots kl}$ and $\hat{p}'_{ij\dots kl}$, respectively. Consequently,

$$-2\log\lambda \sim \log\frac{|\,\hat{A}\,|}{|\,\hat{A}'\,|} + [n-\hat{\mu}']\hat{A}'(n-\hat{\mu}') - [n-\hat{\mu}]\hat{A}(n-\hat{\mu}). \tag{6}$$

When H_0 is true, $\hat{p}_{ij\dots kl}$ and $\hat{p}'_{ij\dots kl}$ both converge stochastically to $p_{ij\dots kl}$. Since it is easily seen that $|\,\hat{A}\,|$ converges stochastically to $|\,A\,|$, it follows that $|\,\hat{A}\,|$ and $|\,\hat{A}'\,|$ converge to the same value; hence (6) reduces to

$$-2\log\lambda \sim [n-\hat{\mu}']\hat{A}'(n-\hat{\mu}') - [n-\hat{\mu}]\hat{A}(n-\hat{\mu}). \tag{7}$$

That the second quadratic form in (7) vanishes may be shown in the following manner. The typical term in the vector $[n-\mu]$ will be denoted by $n_{ij\dots kl}-\mu_{ij\dots kl}$. Since

$$\mu_{ij\dots kl} = E(n_{ij\dots kl}) = nP_{ij\dots k}p_{ij\dots kl},$$

where $P_{ij\dots k}$ denotes the absolute probability of obtaining the $r-1$ chain state $ij\dots k$, it follows that

$$\hat{\mu}_{ij\dots kl} = n\hat{P}_{ij\dots k}\hat{p}_{ij\dots kl}. \tag{8}$$

$P_{ij\dots k}$ is some function of the transition probabilities $p_{ij\dots kl}$. This relationship will be indicated by writing $P_{ij\dots k} = g(p_{ij\dots kl})$. Then by $\hat{P}_{ij\dots k}$ is understood the value

$$\hat{P}_{ij\dots k} = g(\hat{p}_{ij\dots kl}).$$

Assume for the present that the value $g(\hat{p}_{ij\dots kl})$ reduces to

$$g(\hat{p}_{ij\dots kl}) = \frac{n_{ij\dots k}}{n}. \tag{9}$$

Then (8) reduces to $$\hat{\mu}_{ij\dots kl} = n\frac{n_{ij\dots k}}{n}\frac{n_{ij\dots kl}}{n_{ij\dots k}} = n_{ij\dots kl}.$$

The typical term in the vector $[n-\hat{\mu}]$, which is $n_{ij\dots kl}-\hat{\mu}_{ij\dots kl}$, therefore vanishes. As a result, the quadratic form $[n-\hat{\mu}]\hat{A}(n-\hat{\mu})$ vanishes and (7) reduces to

$$-2\log\lambda \sim [n-\hat{\mu}']\hat{A}'(n-\hat{\mu}'). \tag{10}$$

Now
$$\begin{aligned}\mu'_{ij\dots kl} &= nP'_{ij\dots k}p'_{ij\dots kl}\\ &= ng(p_{j\dots kl})\,p_{j\dots kl}.\end{aligned}$$

Consequently,
$$\hat{\mu}'_{ij\dots kl} = ng(\hat{p}_{j\dots kl})\,\hat{p}_{j\dots kl}.$$

As in the case of (9), assume for the present that $g(\hat{p}_{j\dots kl})$ reduces to

$$g(\hat{p}_{j\dots kl}) = \frac{n_{ij\dots k}}{n}. \tag{11}$$

Then $$\hat{\mu}'_{ij\dots kl} = \frac{n_{ij\dots k} n_{j\dots kl}}{n_{j\dots k}},$$

and therefore the typical term of the vector $[n - \hat{\mu}']$ in (10) becomes

$$z_{ij\dots kl} = n_{ij\dots kl} - \frac{n_{ij\dots k} n_{j\dots kl}}{n_{j\dots k}}. \tag{12}$$

It is necessary to determine the number of linearly independent variables of the type (12) in (10) before the distribution of (10) can be determined. Since there are $s^r - 1$ independent restrictions of the type

$$\sum_{i=1}^{s} n_{ij\dots kl} = n_{j\dots kl} = \sum_{l=1}^{s} n_{j\dots kl}$$

and the additional restriction $\sum_{i,\dots,l} n_{ij\dots kl} = n$, there will be only $s^{r+1} - s^r$ linearly independent variables in the original quadratic form in (5). The missing variables may be chosen to be the $n_{ij\dots kl}$ with $l = s$. The variables $z_{ij\dots kl}$ in (12) therefore do not include $l = s$. But from (12) it is clear that $\sum_{i=l}^{s} z_{ij\dots kl} = 0$. Since these restrictions on the $z_{ij\dots kl}$ hold for all values of $j, \dots, k$, but only for $l = 1, \dots, s-1$, there are $s^{r-1}(s-1)$ restrictions on the $z_{ij\dots kl}$. Thus, the quadratic form in (10) can be expressed as a quadratic form in

$$s^{r+1} - s^r - s^{r-1}(s-1) = s^{r-1}(s-1)^2$$

variables.

Since $n_{ij\dots k}/n_{j\dots k}$ converges stochastically to $p_{ij\dots k}$, the variables $z_{ij\dots kl}$ given by (12) will possess the same asymptotic distribution (Cramér, 1946, p. 254) as the variables

$$w_{ij\dots kl} = n_{ij\dots kl} - p_{ij\dots k} n_{j\dots kl}.$$

But since $w_{ij\dots kl}$ is a linear combination of the $n_{ij\dots kl}$, the variables $w_{ij\dots kl}$ will be asymptotically normally distributed. It is easily seen that they have zero means. The quadratic form in (10) therefore has the same asymptotic distribution (Mann & Wald, 1943) as the quadratic form

$$[w]\, B(w), \tag{13}$$

where B denotes the positive definite matrix corresponding to A in (5) after the linearly dependent variables in (12) have been eliminated. Since (13) is a quadratic form in $s^{r-1}(s-1)^2$ linearly independent normal variables with zero means and the matrix B is their asymptotic covariance matrix, (13) will possess a χ^2 distribution with $s^{r-1}(s-1)^2$ degrees of freedom. As a result, the quadratic form in (10) will possess an asymptotic χ^2 distribution with this number of degrees of freedom.

In order to apply the test it is merely necessary to use (1), (2), (3) and (4) to calculate

$$-2\log\lambda = 2 \sum_{i,\dots,l} n_{ij\dots kl} \left(\log \frac{n_{ij\dots kl}}{n_{ij\dots k}} - \log \frac{n_{j\dots kl}}{n_{j\dots k}} \right)$$

and treat $-2\log\lambda$ as a χ^2 variable with $s^{r-1}(s-1)^2$ degrees of freedom.

3. Absolute probability estimates

In the preceding derivation, (9) and (11) were assumed to be true. The verification of the correctness of these intuitive estimates can be made as follows.

Since an r-dependent chain can be expressed as a simple chain, it will suffice to prove (9) and (11) for such chains.

The relationship between absolute probabilities and transition probabilities discussed just after (8) can be expressed by means of equations such as those given by Fréchet (1938). These equations may be written in the form

$$\sum_u P_u p_{uv} = P_v. \tag{14}$$

In the reduction of an r-dependent chain to a simple chain, the indices u and v denote $r-1$ chain states, such as $jk \dots l$. The transition probability p_{uv} will therefore vanish unless the last $r-1$ indices in u are identical with the first $r-1$ indices in v. For v fixed, say $v = jk \dots l$, (14) will then reduce to

$$\sum_{i=1}^{s} P_{ij \dots kl} p_{ij \dots kl} = P_{j \dots kl}. \tag{15}$$

Replacing $p_{ij \dots kl}$ by $\hat{p}_{ij \dots kl}$, it follows that

$$\sum_{i=1}^{s} \hat{P}_{ij \dots k} \frac{n_{ij \dots kl}}{n_{ij \dots k}} = \hat{P}_{j \dots kl}. \tag{16}$$

It is easily seen that (9) satisfies (16), and that the solution is unique.

In order to verify (11), it suffices to place primes on the p's and P's in (15) and follow the same procedure. Since $p'_{ij \dots kl} = p_{j \dots kl}$, (16) will now assume the form

$$\sum_{i=1}^{s} \hat{P}'_{ij \dots k} \frac{n_{j \dots kl}}{n_{j \dots k}} = \hat{P}'_{j \dots kl}.$$

These equations are easily seen to possess the solution given by (11).

REFERENCES

BARTLETT, M. S. (1951). The frequency goodness of fit test for probability chains. *Proc. Camb. Phil. Soc.* **47**, 86.

CRAMÉR, H. (1946). *Mathematical Methods of Statistics*. Princeton University Press.

FRÉCHET, M. (1938). *Recherches Théoriques Modernes sur la Théorie des Probabilités*, 2 livre, p. 37.

MANN, H. B. & WALD, A. (1943). On stochastic limit and order relationships. *Ann. Math. Statist.* **14**, 224.

ON STATIONARY PROCESSES IN THE PLANE

By P. WHITTLE

Applied Mathematics Laboratory, New Zealand Department of Scientific and Industrial Research

The sampling theory of stationary processes in space is not completely analogous to that of stationary time series, due to the fact that the variate of a time series is influenced only by past values, while for a spatial process dependence extends in all directions. This point is elaborated in §§2–4. The estimation and test theory developed in §7 is applied in §8 to uniformity data for wheat and oranges. The final section is devoted to an examination of some particular two-dimensional processes.

1. Introduction

The disturbing effect of topographic correlation on the results of field experiments, forest and crop surveys, sampling surveys of populated areas, etc., is well known, and it is recognized that we have here examples of two-dimensional stochastic processes. The physicists also encounter higher dimensional processes (for instance, in the studies of turbulence and of systems of particles) and have indeed been the principal investigators of the subject.

The processes we mentioned can only as a first approximation be regarded as stationary, if they can be so regarded at all. However, the approximation is satisfactory sufficiently often to make the study of the stationary type of process worth while.

Much of the two-dimensional theory is no more than a formal extension of that used in the study of time series, so we shall treat these aspects as briefly as possible. There is one interesting new feature, however, which is mentioned in the summary above, and we shall consider it at length.

The difficulty of examining particular models is in general greater in two dimensions than in one, simply because there are more mathematical obstacles of a technical nature. One seems to have left the domain of the elementary functions entirely. We shall indicate these obstacles, but take advantage of the fact that for many purposes they may be avoided.

For many applications it is sufficient to consider only purely non-deterministic processes, and we shall restrict our attention to processes of this type, more particularly to linear autoregressions.

2. The line transect

It is useful to begin with a discussion of the simple *line transect* (i.e. a straight line laid over an area, along which observations are taken equidistantly). The observations of the transect may be regarded as being generated by a one-dimensional process, just as are the terms of a time series. However, there is an important difference between the two cases. At any instant in a time series we have the natural distinction of past and future, and the value of the observation at that instant depends only upon *past* values. That is, the dependence extends only in one direction: backwards. In the case of the transect, however, there is no such day and night distinction between the two directions, and dependence will extend both ways. We can consider an example for the more general two-dimensional case of a field: a dab of fertilizer applied at any point in the field will ultimately affect soil fertility in *all* directions. (Exceptions are of course possible; the field may have so strong a slope that only the area on the downward side of the spot will be affected.)

Denoting now observation and 'error' variates by ξ_t, ϵ_t respectively ($t = \ldots, -2, -1, 0, 1, 2, \ldots$), the simplest realistic time series model is perhaps the first-order autoregression

$$\xi_t = a\xi_{t-1} + \epsilon_t. \tag{1}$$

For the transect, however, model (1) would have to be regarded as constituting a degenerate case, in which dependence extended only in one direction. The simplest non-degenerate transect model would be

$$\xi_t = a\xi_{t-1} + b\xi_{t+1} + \epsilon_t, \tag{2}$$

where it is intuitively clear that a and b must not be too large. We shall term (2) a bilateral autoregression, in distinction to (1) which is a unilateral autoregression.

Consequences of the admission of the bilateral type of scheme become apparent when one considers the estimation of parameters. Whereas the parameter a in (1) may be estimated consistently by minimizing the residual sum of squares $\sum_t (\xi_t - a\xi_{t-1})^2$, an attempt to estimate a and b in (2) simply by minimizing

$$U = \sum_t (\xi_t - a\xi_{t-1} - b\xi_{t+1})^2, \tag{3}$$

leads to nonsensical results. One could explain the breakdown by saying that it is not legitimate to include ξ_{t+1} in the conditional mean of ξ_t when the value of ξ_{t+1} itself depends upon ξ_t. Formally, we may say that the Jacobian of the transformation from the ϵ_t to the ξ_t is not unity for relation (2) as it would be for a unilateral relation such as (1). It may be shown, however (and in § 7 will be proved for the more general two dimensional case), that the correct equations for the least-square estimates are obtained by minimizing kU, where k is a certain function of the parameters given by

$$\log k = -\frac{1}{2\pi}\int_0^{2\pi} \log\left(a\,e^{i\omega} - 1 + b\,e^{-i\omega}\right)\left(a\,e^{-i\omega} - 1 + b\,e^{i\omega}\right) d\omega. \tag{4}$$

3. FORMAL PROPERTIES OF THE BILATERAL SCHEME

We shall briefly examine the properties of the transect model as a preparation for the discussion of the two-dimensional case. Let us consider the general bilateral linear autoregression

$$L(T)\,\xi_t = \epsilon_t, \tag{5}$$

where $L(T) = \Sigma a_j T^j$ and T is the translation operator

$$T\xi_t = \xi_{t+1}. \tag{6}$$

Equation (5) has the solution

$$\xi_t = \frac{\epsilon_t}{L(T)} = \Sigma b_j \epsilon_{t+j}, \tag{7}$$

where b_j is the coefficient of $e^{ij\omega}$ in the Fourier expansion of $[L(e^{i\omega})]^{-1}$ (see Bartlett, 1946, p. 60). Using Fourier transforms it may similarly be shown that the spectral function of scheme (5) is

$$F(\omega) = \frac{\sigma^2(\epsilon)}{L(e^{i\omega})\,L(e^{-i\omega})} \tag{8}$$

(cf. Doob, 1944; Daniell, 1946). The autocovariances

$$\phi(j) = \operatorname{cov}(\xi_t, \xi_{t+j}) \quad (j = 0, \pm 1, \pm 2, \ldots) \tag{9}$$

are generated as the coefficients in the Fourier expansion of $F(\omega)$. Thus, writing for convenience $e^{i\omega} = z$, $\sigma^2(\epsilon) = v$, we have

$$F(\omega) = \frac{v}{L(z)\,L(z^{-1})}. \tag{10}$$

It is as well to consider the conditions under which expressions (7) and (8) are valid, since a number of essential points are involved. If (5) is to represent a unilateral scheme, expressing 'dependence only on the past', then the Laurent expansion of $[L(z)]^{-1}$ on $|z| = 1$ must not involve positive powers of z, and we are led to the usual condition for the stability of a time-series autoregression: that all roots of $L(z) = 0$ fall *inside* the unit circle (Wold, 1938). If, however, one is prepared to admit bilateral schemes, then all that is required is that $[L(z)]^{-1}$ possess a convergent Laurent expansion—coefficients otherwise unrestricted. The stationarity condition is in this case greatly weakened; all that is required is that no root of $L(z) = 0$ fall *on* the unit circle.

While this is the only condition which need be imposed upon a particular $L(z)$ to secure stationarity, it is desirable that $L(z)$ be such that in relation (5) it is ξ_t which is the dependent variate (not, for example, ξ_{t+1}, as would be the case if relation (1) were modified to $a\xi_t = \xi_{t+1} - \epsilon_t$). This may be achieved by multiplying $L(z)$ by some integral power of z (or, what is the same thing, redefining the ϵ sequence by translation so that ϵ_t comes into correspondence with ξ_t). For, suppose that as z describes the unit circle once in a positive direction $L(z)$ encircles the origin in the complex plane r times (where r is necessarily integral). The normalization will then be achieved by translating the ϵ sequence r steps backwards, so that $L(T)$ is replaced by the normalized operator $T^{-r}L(T)$, or $L^*(T)$, say. Since the origin is not a branch point for $\log L^*(z)$ (as it was for $\log L(z)$ if r was not zero) then $\log L^*(z)$ may be expanded in a Laurent series in z, and the operator $L^*(T)$ may be represented

$$L^*(T) = e^{\Sigma c_j T^j}. \tag{11}$$

It will be assumed in the sequel that this normalization has been performed, so that the representation (11) is always possible.

4. Indeterminacies of estimation

If one is prepared to admit bilateral schemes, then the autoregression exhibits the same kind of ambiguity that Wold (1938) has shown to obtain for the moving average. Thus, suppose that one is given a set of autocovariances which could have been generated by an autoregression of order p, and that one is set the task of determining the autoregression, i.e. of determining the polynomial $L(T)$ of (5). It is thus necessary to find an $L(z)$ which will satisfy equation (10), where $F(\omega)$ is determined by the given autocovariances. Such an $L(z)$ may be chosen in anything up to 2^p ways, since if a particular $L(z)$ has roots $\alpha_1, \alpha_2, \ldots, \alpha_p$, then there are 2^p possible finite autoregressions, corresponding to the 2^p possible representations of $F(\omega)$:

$$F(\omega) = \frac{\text{const.}}{|(z^{\pm 1} - \alpha_1)(z^{\pm 1} - \alpha_2)\ldots(z^{\pm 1} - \alpha_p)|^2}. \tag{12}$$

Two of these schemes are unilateral, and differ only in that the 'time' axis runs in opposite directions.

A similar argument applies for the least square fitting of a pth order autoregression to a set of data, since here again one works from the autocovariances, which could as easily have stemmed from one of the 2^p fitted autoregressions as from another.

For the sake of example, let us examine once more the bilateral scheme (2). If the scheme has been normalized according to the last paragraph of § 3 then the equation

$$a - z + bz^2 = 0 \tag{13}$$

will have one root inside the unit circle and one outside. Let us denote the roots α, β^{-1}, $[|\alpha| < 1, |\beta| < 1]$. Defining coefficients A and B by the relation

$$(z-\alpha)(z-\beta) = z^2 + Az + B, \tag{14}$$

we can construct an autoregression

$$\xi_t + A\xi_{t-1} + B\xi_{t-2} = \epsilon_t, \tag{15}$$

which is unilateral and yet generates the same autocorrelations as (2).

Evaluating integral (4) we find that a and b will be estimated by minimizing the quantity

$$\begin{aligned}[1+\sqrt{(1-4ab)}]^{-2}\Sigma(\xi_t - a\xi_{t-1} - b\xi_{t-2})^2 \\ \approx [1+\sqrt{(1-4ab)}]^{-2}[(1+a^2+b^2)C_0 - 2(a+b)C_1 + 2abC_2],\end{aligned} \tag{16}$$

where C_s is the observed autocovariance of lag s. Transforming to parameters A and B we find expression (16) proportional to

$$(1+A^2+B^2)C_0 + 2(A+AB)C_1 + 2BC_2 \approx \text{const.}\ \Sigma(\xi_t + A\xi_{t-1} + B\xi_{t-2})^2, \tag{17}$$

which is another indication of the equivalence of schemes (2) and (15). The practical significance of this equivalence is that the minimization of the rather awkward expression (16) may be replaced by that of the much simpler expression (17). The estimates of a and b may then be calculated from those of A and B by means of relation (14).

It may seem unnecessary to introduce the bilateral type of scheme when any such scheme may effectively be reduced to a unilateral one. Indeed, the step appears positively undesirable when we consider how much more complicated and indeterminate the parameter estimates are for a bilateral model. We shall see, however, that in the two-dimensional case the reduction to a unilateral scheme complicates matters very much, in contrast to the one-dimensional case we have just considered. There is thus no escaping the explicit introduction of a dependence in all directions. A further point is relevant: it is the multilateral scheme which in general corresponds to reality, even in those cases for which the formal work of estimation, etc., is simplest performed using an equivalent unilateral model.

5. GENERALITIES ON THE TWO-DIMENSIONAL PROCESS

One can consider continuous processes in which the variate has a value at every point of the plane, or discrete processes, which are usually such that the variate is observed only at the points of a rectangular plane lattice. We shall for the most part consider the discrete case, as being simpler mathematically and of greater practical interest. The observed and error variates will be denoted ξ_{st} and ϵ_{st} respectively ($s, t = \ldots, -2, -1, 0, 1, 2, \ldots$). When discussing continuous processes this notation will be modified to $\xi(x, y)$ and $\epsilon(x, y)$ (x and y assuming all real values).

The particular model which we shall consider almost exclusively is the two-dimensional linear autoregression, which we shall write

$$L(T_s, T_t)\xi_{st} = \epsilon_{st}, \tag{18}$$

where T_s and T_t are translation operators defined by

$$T_s\xi_{st} = \xi_{s+1,t}, \quad T_t\xi_{st} = \xi_{s,t+1} \tag{19}$$

and

$$L(T_s, T_t) = \sum_j \sum_k a_{jk} T_s^j T_t^k. \tag{20}$$

Corresponding to equations (7)–(10) we have

$$\xi_{st} = \frac{\epsilon_{st}}{L(T_s, T_t)} = \sum_j \sum_k b_{jk} \epsilon_{s+j,t+k}, \tag{21}$$

$$F(\omega_1, \omega_2) = \frac{v}{L(z_1, z_2)\, L(z_1^{-1}, z_2^{-1})} \tag{22}$$

$$= \sum_j \sum_k \phi(j,k) z_1^j z_1^k, \tag{23}$$

where b_{jk} is the coefficient of $z_1^j z_2^k$ in the Fourier expansion of $[L(z_1, z_2)]^{-1}$, ($z_1 = e^{i\omega_1}, z_2 = e^{i\omega_2}$), v is the variance of the ϵ's, $\phi(j,k)$ the covariance of ξ_{st} and $\xi_{s+j,t+k}$, and $F(\omega_1, \omega_2)$ the corresponding spectral function, as defined by (23).

As before, it is necessary and sufficient for the validity of (21) and (22) that $L(z_1, z_2)$ be not zero for any z_1 and z_2 which simultaneously satisfy $|z_1| = 1, |z_2| = 1$ (at least when the autoregression (18) is finite). We shall assume that the operator L has been so normalized that $L(z_1, z_2)$ will not circle the origin in the complex plane as either z_1 or z_2 moves around the unit circle. (If L does not circle the origin as z_1 traces out the unit circle and z_2 holds a particular value, neither will it do so for any other value of z_2. For if it should do so there would be some intermediate value of z_2 for which L would move *through* the origin, against hypothesis.)

6. Unilateral representation of the two-dimensional process

We shall prove, by a type of argument which is now well known (due to Wiener; see, for example, Wiener, 1949, p. 78), that a unique process may be found which will generate a given set of autocorrelations and in which ξ_{st} is expressed as an autoregression upon ξ_{su} ($u > t$) and ξ_{vw} ($v > s, w$ unrestricted). That is, in the lattice of Fig. 1 the value at the white dot may be expressed in terms of the values at the black dots. The given autocorrelogram may obviously not be completely arbitrary, but the necessary conditions are very mild.

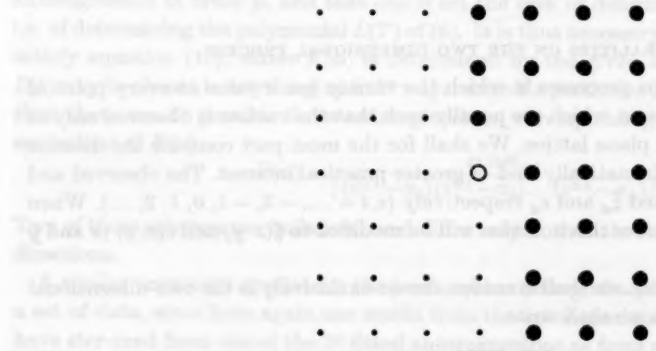


Fig. 1

The first condition is that the autocorrelations be such that they could have been generated by a purely non-deterministic process, which implies among other things that $\log F(\omega_1, \omega_2)$ has a Fourier expansion, so that $F(\omega_1, \omega_2)$ may be represented

$$F(\omega_1, \omega_2) = \exp\left[\sum_j \sum_k \alpha_{jk} z_1^j z_2^k\right]. \tag{24}$$

Define now the function

$$P(z_1, z_2) = \exp - \left[\frac{\alpha_{00}}{2} + \sum_{k=1}^{\infty} \alpha_{0k} z_2^k + \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_{jk} z_1^j z_2^k\right]. \tag{25}$$

Since

$$F(\omega_1, \omega_2) = \frac{1}{P(z_1, z_2)\, P(z_1^{-1}, z_2^{-1})}, \tag{26}$$

the autoregression

$$P(T_s, T_t)\, \xi_{st} = \epsilon'_{st} \quad (\sigma^2(\epsilon') = 1) \tag{27}$$

will have spectral function $F(\omega_1, \omega_2)$, as required. Further, the expansion of $P(T_s, T_t)$ will involve the same powers of T_s and T_t as occur of z_1 and z_2 in expression (25), so that the autoregression is of the required form. To ensure that (27) may validly be written out as an autoregression we include a second condition: that $P(e^{i\omega_1}, e^{i\omega_2})$ possess a Fourier expansion.

Representation (27) corresponds to the unilateral representation in the case of a transect. It is by no means as useful, however, and for such purposes as estimation one is in general better advised to work with the original model. For one thing, it is not true in two dimensions as it is in one, that the unilateral representation of a finite autoregression is also a finite autoregression. This is confirmed by the following simple example: the finite autoregression

$$(1+\beta^2)\, \xi_{st} = \beta(\xi_{s+1,t} + \xi_{s,t+1} + \xi_{s,t-1}) + \epsilon_{st} \tag{28}$$

has a unilateral representation which is infinite:

$$\xi_{st} = 2\beta \xi_{s,t+1} - \beta^2 \xi_{s,t+2} - \beta^2 \xi_{s+1,t+1} + \beta(1-\beta^2) \sum_0^{\infty} \beta^j \xi_{s+1,t-j} + \epsilon'_{st}. \tag{29}$$

Further, the real usefulness of the unilateral representation is that it suggests a simplifying change of parameters (although the same parameter transformation is usually suggested in the evaluation of integral (4) or its two-dimensional equivalent). For most two-dimensional models, however, the appropriate transformation, even if evident, is so complicated that nothing is gained by performing it. For example, to calculate the unilateral representation of the model

$$\xi_{st} = \alpha(\xi_{s+1,t} + \xi_{s-1,t} + \xi_{s,t+1} + \xi_{s,t-1}) + \epsilon_{st} \tag{30}$$

one must evaluate the coefficients in the Fourier expansion of $\log[1 - \alpha(z_1 + z_1^{-1} + z_2 + z_2^{-1})]$, and these are not expressible in terms of anything simpler than elliptic integrals.

7. SAMPLING THEORY

In this section we shall consider the sampling theory of the discrete process, deriving general estimation equations, tests of fit, and formulae for the asymptotic variances and covariances of parameter estimates. The pattern of development is similar to that of previous articles on time-series analysis (Whittle, 1951–3), apart from the fact that special attention must be paid to the effect of multilaterality. The particular processes considered are those which are purely non-deterministic and for which $[F(\omega_1, \omega_2)]^{-1}$ is not zero for any real ω_1, ω_2, so that the process is representable as an autoregression. We shall also in general assume the

variates normally distributed. Estimation equations thus obtained on the maximum-likelihood criterion can be regarded as least-square estimation equations when the variate distribution is not normal, but the extent to which remaining results are valid in such a case is a matter for further inquiry.

Let us suppose that we have a series of mn observations ξ_{st} $(s = 1, 2, \ldots, m;\ t = 1, 2, \ldots, n)$, so that the empirical covariance of lag j, k is

$$C_{jk} = \frac{1}{mn} \sum_{s=1}^{m-j} \sum_{t=1}^{n-k} \xi_{st} \xi_{s+j,t+k}. \tag{31}$$

We shall uniformly neglect end-effects, and so have divided in (31) by mn instead of $(m-j)(n-k)$. Let us further define the quantity

$$\begin{aligned} f(\omega_1, \omega_2) &= \frac{1}{mn} \{[\Sigma\Sigma \xi_{st} \cos(s\omega_1 + t\omega_2)]^2 + [\Sigma\Sigma \xi_{st} \sin(s\omega_1 + t\omega_2)]^2\} \\ &= \sum_{-m}^{m} \sum_{-n}^{n} C_{jk} \cos(j\omega_1 + k\omega_2) \\ &= \sum_{-m}^{m} \sum_{-n}^{n} C_{jk} e^{i(j\omega_1 + k\omega_2)}, \end{aligned} \tag{32}$$

which we see on comparison with (23) to be the *empirical spectral function*, and in fact the appropriate extension of the Schuster periodogram to two dimensions.

At the end of the section we shall prove the following basic result: that if the variates ξ_{st} are normally distributed with zero mean, and are generated by a stationary process of the type considered with spectral function $F(\omega_1, \omega_2)$, then their joint likelihood is given, apart from end-effects, by the expression

$$p(\xi) = \frac{1}{(2\pi V)^{\frac{1}{2}(mn)}} \exp\left[-\frac{mn}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f}{F} d\omega_1 d\omega_2 \right], \tag{33}$$

where

$$V = \exp\left[\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log F \, d\omega_1 d\omega_2 \right]. \tag{34}$$

Taking the logarithm of (33) we see that the maximum-likelihood estimates of the parameters of F are obtained by minimizing

$$\mathfrak{L} = \frac{1}{4\pi^2} \iint \left[\log F + \frac{f}{F} \right] d\omega_1 d\omega_2. \tag{35}$$

Even if the variates are not considered to be normally distributed, expression (35) will upon minimization yield the least-square estimates (see Whittle, 1953, p. 132).

Expression (35) actually does not lead to the most convenient form of the estimation equations, this is obtained by expressing $\mathfrak{L}$ in terms of the autocovariances rather than the periodogram

$$\mathfrak{L} = \frac{1}{4\pi^2} \iint \log F \, d\omega_1 d\omega_2 + \Sigma\Sigma c_{jk} C_{jk}, \tag{36}$$

where the coefficients c are given by

$$\Sigma\Sigma c_{jk} z_1^j z_2^k = \frac{1}{F(\omega_1, \omega_2)}. \tag{37}$$

Consider now the special case of a process generated by a stochastic difference equation of type (18) so that

$$F = \frac{v}{L(z_1, z_2)\, L(z_1^{-1}, z_2^{-1})}. \tag{38}$$

Inserting this relation in (36) we obtain

$$\mathfrak{L} = \log v + \log k + U/mnv, \tag{39}$$

where

$$U = \sum_1^m \sum_1^n \epsilon_{st}^2 = \sum_1^m \sum_1^n [L(T_s, T_t)\, \xi_{st}]^2 = \sum_j \sum_k c_{jk} C_{jk} \tag{40}$$

and

$$\log k = -\frac{1}{4\pi^2} \iint \log [L(z_1, z_2)\, L(z_1^{-1}, z_2^{-1})]\, d\omega_1 d\omega_2. \tag{41}$$

The minimized value of (39) with respect to the irrelevant parameter v is $\log (kU/mn)$, so that the quantity to be minimized with respect to the parameters of L is kU. This is the result which is of most use in investigations of the present type: that the least-square estimates are yielded by the minimization of the usual 'residual sum of squares' times a function of the parameters, k. We see from (41) that $-\frac{1}{2}\log k$ may be interpreted as the absolute term in the double Fourier expansion of $L(e^{i\omega_1}, e^{i\omega_2})$, a fact that may sometimes ease its evaluation.

Suppose that F contains the unknown parameters $\theta_1, \theta_2, \ldots, \theta_p$ (v being included among these, if, as is usual, its value is unknown). Then, following a familar line of reasoning (see Whittle, 1953, p. 135, for an application similar to the present one) that the asymptotic covariance matrix of the least square estimates of $\theta_1, \theta_2, \ldots, \theta_p$ is

$$\frac{2}{mn}\left[\frac{1}{4\pi^2}\iint \frac{\partial \log F}{\partial \theta_j}\frac{\partial \log F}{\partial \theta_k}\, d\omega_1 d\omega_2\right]^{-1}. \tag{42}$$

Suppose again that when the p parameters have been fitted the minimized value of kU is $(kU)_p$, and that when an additional q parameters have been fitted the quantity falls to $(kU)_{p+q}$. Then we can test the improvement attained by the introduction of the extra q parameters by using the fact that, if the initial p-parameter hypothesis were correct, then

$$\psi^2 = (mn - p - q) \log \frac{(kU)_p}{(kU)_{p+q}} \tag{43}$$

would be asymptotically distributed as χ^2 with q degrees of freedom. The argument is again an old one (cf. Whittle, 1952, 1953).

Proof of relation (33). Let the process under consideration have the unilateral representation (27). The joint frequency function of the mn residuals ϵ'_{st} ($s = 1, 2, \ldots, m; t = 1, 2, \ldots, n$) is

$$p(\epsilon') = \frac{1}{(2\pi)^{\frac{1}{2}(mn)}} \exp\left[-\frac{1}{2}\sum_1^m \sum_1^n (\epsilon'_{st})^2\right]. \tag{44}$$

Performing the linear transformation to which (27) is equivalent, we obtain (neglecting end effects) the following expression for the frequency function of the ξ_{st}:

$$\begin{aligned} p(\xi) &\approx \frac{A^{mn}}{(2\pi)^{\frac{1}{2}(mn)}} \exp\left[-\frac{1}{2}\sum_1^m \sum_1^n [P(T_s, T_t)\, \xi_{st}]^2\right] \\ &\approx \frac{A^{mn}}{(2\pi)^{\frac{1}{2}(mn)}} \exp\left[-\frac{mn}{2}\sum\sum c_{jk} C_{jk}\right] \\ &\approx \frac{A^{mn}}{(2\pi)^{\frac{1}{2}(mn)}} \exp\left[-\frac{mn}{8\pi^2}\iint \frac{f}{F}\, d\omega_1 d\omega_2\right], \end{aligned} \tag{45}$$

where A is the coefficient of ξ_{st} in (27). By (24) and (25)

$$A = e^{-\frac{1}{2}\alpha_{00}} = \exp\left[-\frac{1}{8\pi^2}\iint \log F \, d\omega_1 d\omega_2\right], \tag{46}$$

which on insertion in (45) leads to (33).

The nature of the approximate equality in (45) may be specified more exactly. If the right-hand member in (45) is denoted p', then the relation obtaining is that $\log p$ and $\log p'$ are asymptotically equal (neglect of end-effects leading to neglect of terms in the exponent). This kind of approximation is quite satisfactory, since the addition of a term of relative order m^{-1} or n^{-1} to the logarithm of the likelihood will for large m and n have little effect on either the values of the maximum-likelihood estimates, or the significance points of likelihood ratios.

8. Numerical examples

Neither of the two sets of numerical data to be examined provides a picture book example of the particular models we have been studying, but they are perhaps the more valuable for just that reason.

The first set of data concern a uniformity trial on wheat (yield of grain) conducted by Mercer & Hall (1911). It involves 500 plots, each 11 ft by 10·82 ft., arranged in a 20×25 rectangle, plot totals constituting the observations. The first and fourth quadrants of the correlation field are given in Table 1; the second and third quadrants can be filled in by means of the relation

$$\phi(-j, -k) = \phi(j, k).$$

We note that correlations along the north-south (s) axis are considerably stronger than those along the east-west (t) axis, at least part of the explanation presumably being that the plots are not square. Correlations are generally higher in the north-east direction than in the north-west, indicating a definite directional effect. Another fact worthy of note is that correlations do not decrease monotonically as one moves out from the origin, instead they dip to a minimum and then rise again. One could imagine competition between neighbouring plants producing such an effect if the distances were smaller, but the more likely explanation in the present case is that there are 'waves of fertility' of the kind often remarked in a ploughed field (see, for example, Neyman, 1952, p. 75).

In Table 2 we summarize the various schemes considered, giving the fitted coefficients and the corresponding values of k, U and kU.

Since all kU are near to 0·7 we can say roughly that if the introduction of a new parameter depresses kU by Δ, then by equation (43) Δ is significant at the 5 % level if

$$3{\cdot}841 < 500 \log\left(\frac{0{\cdot}7 + \Delta}{0{\cdot}7}\right)$$

or $\Delta > 0{\cdot}0054$. The corresponding limits if two or three parameters have been introduced are 0·0084 and 0·0110. These limits provide useful rules of thumb for judging significances of differences in kU in the table. Thus, hypotheses 1 and 3 do not differ significantly in degree of fit, while 1 and 4 very definitely do. Hypotheses such as 1 and 5 cannot strictly be compared in this manner, since neither hypothesis is a special instance of the other. However, one would be very much inclined to say that the fit of 1 is superior to that of 5, and this conclusion is borne out if one compares each in turn with the more general hypothesis 7.

The surprising feature of the results is that the simple unilateral scheme 1 fits the data so much better than the symmetric second order schemes 5 and 6. Thus, although the theme of this article has largely been that spatial processes will in general be multilateral, our first example appears to be dominantly unilateral. The reason for this becomes apparent upon a re-examination of the correlations of Table 1. These correlations fall steeply as soon as a lag is introduced (to 0·52 on the s axis and 0·29 on the t axis), but decrease only slowly as the lag is increased (the corresponding figures for a lag of two units are thus 0·41 and 0·15). However, it will be seen in § 9 that the correlogram of a scheme such as 5 or 6 decays smoothly right from the origin, having in fact zero derivative there (cf. Fig. 2), so that neither 5 nor 6 can possibly fit the observations well. There are at least two possible explanations of this behaviour of the observed correlogram. The underlying scheme may in fact be a unilateral

Table 1. *Autocorrelations for the wheat data, $s = 0$ to 4, $t = -3$ to $+3$*

t	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
−3	0·1880	0·1602	0·1509	0·1276	0·1352
−2	0·1510	0·0234	0·0020	−0·0137	−0·1039
−1	0·2923	0·1853	0·1349	0·0788	0·0878
0	1·0000	0·5252	0·4055	0·3639	0·3561
1	0·2923	0·2354	0·1799	0·1205	0·1399
2	0·1510	0·1285	0·0999	0·0749	0·0859
3	0·1880	0·1935	0·2483	0·2415	0·2284

Table 2. *Details of the models fitted*

Model no.	$L(T_s, T_t)$	k	U	kU
1	$1-0{\cdot}488T_s-0{\cdot}202T_t$	1	0·6848	0·6848
2	$1-0{\cdot}483T_s-0{\cdot}179T_t^{-1}$	1	0·6940	0·6940
3	$1-0{\cdot}492T_s-0{\cdot}211T_t+0{\cdot}019T_sT_t$	1	0·6845	0·6845
4	$1-0{\cdot}402T_s-0{\cdot}168T_t-0{\cdot}172T_s^2-0{\cdot}092T_t^2$	1	0·6564	0·6564
5	$1-0{\cdot}159(T_s+T_s^{-1}+T_t+T_t^{-1})$	1·1240	0·6508	0·7314
6	$1-0{\cdot}213(T_s+T_s^{-1})-0{\cdot}102(T_t+T_t^{-1})$	1·1332	0·6217	0·7045
7	$1-0{\cdot}488T_s+0{\cdot}030T_s^{-1}-0{\cdot}202T_t-0{\cdot}034T_t^{-1}$	0·9843	0·6816	0·6709

one, due to e.g. the presence of a slope in the ground or a prevailing wind. Alternatively, we must recall that the observations are not *point* observations of growth, but *integrated* observations of the growth over an area. An integration such as this will enhance the autocovariance of zero lag relative to the others, and the correlogram of a scheme such as 6, when thus distorted, would not be dissimilar to the observed correlogram. However, the consideration of such an effect would lead us too far for present purposes, and we shall take the observations at their face value.

Scheme 1 is not completely satisfactory, in that it does not explain the dip and rise in the correlation for increasing distance. The inclusion of second-order terms in the same direction as the first-order terms improves the fit considerably, see scheme 4.

Unilateral schemes such as 1–4 can be fitted directly by least square regression methods, since for them $k = 1$. The fitting of multilateral schemes such as 5–7 is more difficult, however. The evaluation of U according to (40) is direct, it is the evaluation of k from (41) which presents new problems. For the present calculations the pedestrian methods of series expansion and numerical integration have been employed, although it is likely that better methods exist. Thus, for a scheme

$$\xi_{st} = \alpha\xi_{s+1,t} + \beta\xi_{s-1,t} + \gamma\xi_{s,t+1} + \delta\xi_{s,t-1} + \varepsilon_{st}, \tag{47}$$

we have $\log(k) =$ minus twice absolute term in $\log[1 - \alpha z_1 - \beta z_1^{-1} - \gamma z_2 - \delta z_2^{-1}]$

$$= \sum_{j=1}^{\infty} \sum_{k=0}^{j} \frac{(2j)!}{j[k!\,(j-k)!]^2} (\alpha\beta)^k (\gamma\delta)^{j-k}. \tag{48}$$

For $\alpha\beta = \gamma\delta$ this reduces to

$$\log(k) = \sum_{j=1}^{\infty} \frac{1}{j} \binom{2j}{j}^2 (\alpha\beta)^j. \tag{49}$$

Expansions (48) and (49) are useful as long as $\alpha\beta$ and $\gamma\delta$ are very small, but convergence becomes slow as these quantities approach their maximum value of $\frac{1}{16}$. In Table 3 the value of $\log k$ in (49) is given for some specimen values of $\theta = \sqrt{(\alpha\beta)}$; these values have been obtained by numerical integration, and vary slowly enough to permit quite a fair graphical interpolation.

Table 3. *Specimen values of the correcting factor* $\log(k)$

θ	0·00	0·05	0·10	0·15	0·20	0·22	0·25
$\log(k)$	0·0000	0·0076	0·0420	0·1010	0·2028	0·2656	0·4406

Table 4. *Autocorrelations for the orange data,* $s = 0$ to 9, $t = -4$ to 4.

t	Values of s									
	0	1	2	3	4	5	6	7	8	9
−4	0·3912	0·3902	0·3771	0·3581	0·3710	0·2923	0·2549	0·2546	0·2440	0·2131
−3	·4609	·3956	·3930	·3696	·3484	·3353	·3137	·2834	·2894	·2870
−2	·4667	·4159	·3982	·3470	·3243	·3225	·3250	·2801	·2721	·2752
−1	·5462	·4669	·4336	·3854	·3880	·3761	·3495	·2914	·2567	·2606
0	1·0000	·5403	·5052	·4840	·4585	·4233	·3960	·3246	·3210	·3150
1	0·5462	·4458	·4280	·3751	·3914	·3622	·3716	·2948	·2958	·2925
2	·4667	·3858	·3799	·3459	·3603	·3475	·3401	·2905	·2766	·2230
3	·4609	·4190	·3716	·3764	·3509	·3336	·3369	·2765	·2433	·2741
4	·3912	·3543	·3882	·3392	·3537	·3446	·3305	·2893	·2685	·2321

The models of Table 2 were fitted simply by inserting trial values of the parameters and calculating the corresponding kU. Improved values were then found by approximating kU in the neighbourhood of its minimum by a paraboloid, and then locating this approximate minimum, whereupon the process was repeated. The work is tedious, but not unreasonably so if one considers the effort which the data themselves represent.

Our second example is derived from a uniformity trial of 1000 orange trees by Batchelor & Reed (1924), the trees being arranged in a 20×50 rectangular lattice. The correlation field $\rho(s,t)$ is given in Table 4, and the correlations along the s axis plotted in Fig. 2. The correlations in the two quadrants follow a very similar pattern, indicating a high degree of symmetry in the original experimental field. A feature of interest is, that while the correlations $\rho(s,t)$ for the most part change smoothly in value as one moves through adjacent values of s and t, there is a sharp discontinuity at the origin. The correlation surface thus consists of a smoothly rounded broad dome with a spike at the apex. This suggests that the yield of an individual tree, Y_{st}, may be represented

$$Y_{st} = \xi_{st} + \eta_{st}, \tag{50}$$

where ξ_{st} obeys a process with smooth correlogram, while distinct η_{st} are uncorrelated with one another (or with the ξ_{st}) so that the η correlogram consists simply of a 'spike' at the origin. A natural interpretation of (50) springs to mind: that ξ represents the fertility of the

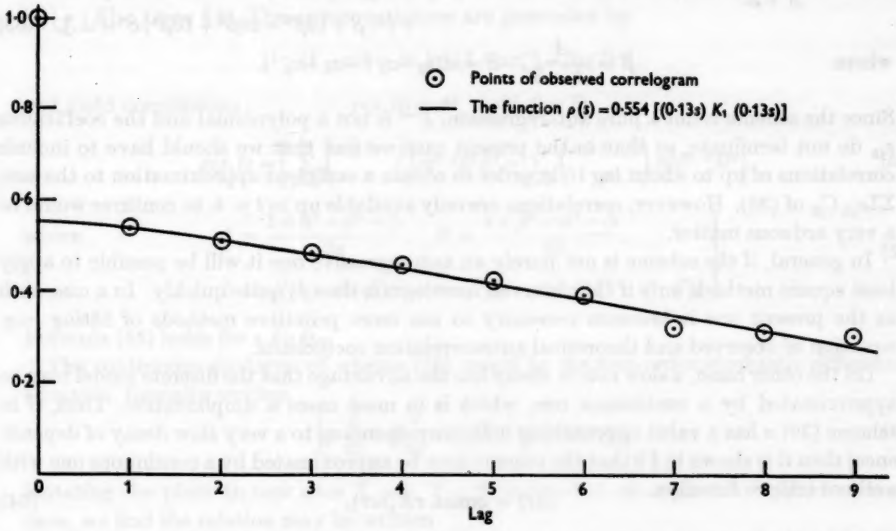


Fig. 2

soil around the tree, and presumably obeys some simple symmetric process such as (30), while η represents the intrinsic fruitfulness of the individual tree. Looking at the correlogram, one would say that soil variability and tree variability contribute roughly 56 and 44 % of the total variation respectively. However, one should not forget the possibility that the tree may 'integrate' the fertility of the surrounding soil as in the previous example.

It is a much more tedious matter to fit a compound scheme such as (50) plus (30) than to fit a simple autoregression such as (30). Supposing $\text{var}(\eta) = A$ and $\text{var}(\epsilon) = B$ we have

$$F(\omega_1, \omega_2) = A + \frac{B}{[1-\alpha(z_1+z_1^{-1}+z_2+z_2^{-1})]^2}. \tag{51}$$

The quantities required for estimation purposes are $\log V$ and the coefficients in the expansion of F^{-1} (see (36)). We find

$$\begin{aligned}\log V = &+\frac{1}{4\pi^2}\iint \log F\, d\omega_1 d\omega_2 = \log(A+B)+\sum_{j=1}^{\infty}\frac{1}{j}\binom{2j}{j}\alpha^{2j}t\\ &+4(\alpha\beta)^2[1-2\beta]\\ &+36(\alpha\beta)^4[-\tfrac{1}{2}+4\beta-4\beta^2]\\ &+400(\alpha\beta)^6[\tfrac{1}{3}-6\beta+16\beta^2-\tfrac{32}{3}\beta^3]\\ &+4900(\alpha\beta)^8[-\tfrac{1}{4}+8\beta-40\beta^2+60\beta^3-32\beta^4]\\ &+63504(\alpha\beta)^{10}[\tfrac{1}{5}-10\beta+80\beta^2-224\beta^3+256\beta^4-\tfrac{1024}{10}\beta^5]\\ &+\ldots,\end{aligned} \tag{52}$$

and

$$\begin{aligned}F^{-1} &= \Sigma\Sigma c_{jk} z_1^j z_2^k\\ &= \frac{1}{A+B}[1+(2\beta-2)S+(1-5\beta+4\beta^2)S^2+(4\beta-12\beta^2+8\beta^3)S^3\\ &\qquad +(-\beta+13\beta^2-28\beta^3+16\beta^4)S^4+\ldots],\end{aligned} \tag{53}$$

where $$\beta = \frac{A}{A+B}, \quad S = \alpha(z_1+z_1^{-1}+z_2+z_2^{-1}).$$

Since the scheme is not a pure autoregression, F^{-1} is not a polynomial and the coefficients c_{jk} do not terminate, so that in the present case we find that we should have to include correlations of up to about lag 10 in order to obtain a sufficient approximation to the sum $\Sigma\Sigma c_{jk}C_{jk}$ of (36). However, correlations are only available up to $t = 4$, to continue would be a very arduous matter.

In general, if the scheme is not purely an autoregressive one it will be possible to apply least square methods only if the observed correlogram decays quite quickly. In a case such as the present one it becomes necessary to use more primitive methods of fitting: e.g. equation of observed and theoretical autocorrelation coefficients.

On the other hand, a slow rate of decay has the advantage that the discrete model may be approximated by a continuous one, which is in most cases a simplification. Thus, if in scheme (30) α has a value approaching 0·25 (corresponding to a very slow decay of dependence) then it is shown in § 9 that the scheme may be approximated by a continuous one with autocovariance function

$$\rho(r) = \text{const.}\, rK_1(\kappa r), \tag{54}$$

where r is the distance between the two points considered, K_1 a modified Bessel function, and

$$\kappa = \sqrt{\left(\frac{1}{\alpha}-4\right)}. \tag{55}$$

The thin line of the correlogram (Fig. 2) indicates the values of function (55) when the constants have been adjusted for coincidence at r_1 and r_8. It is found that $\kappa = 0{\cdot}13$, whence $\alpha = 0{\cdot}2489$.

The agreement looks impressive, but must be discounted considerably, since almost any monotone decreasing function would fit the observed curve reasonably well if only the endpoints were arrange to coincide. However, if one fits, for example, an exponential curve in the same fashion, agreement is not at all as good, the exponential curve sagging too much in the middle.

9. SPECIAL PROCESSES

Strictly, one does not need to be acquainted with the properties of a model in order to fit it to a set of data (cf. the examples of the previous section). However, a knowledge of these properties does help one to arrive at a good model quickly, to say the least. In the present context 'properties' is usually synonymous with 'correlogram', since it is most often the correlogram one uses to characterize a given set of data. We shall now examine a few special models and their correlograms, but in a rather unsystematic fashion, since few results of an explicit nature are obtainable.

The simplest model of interest is the degenerate autoregression

$$\xi_{st} = \alpha\xi_{s+1,t} + \beta\xi_{s,t+1} + \epsilon_{st}, \tag{56}$$

with 'solution'

$$\xi_{st} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j+k}{j}\alpha^j\beta^k\epsilon_{s+j,t+k}. \tag{57}$$

For the process to be stationary, ξ_{st} being taken as the dependent variate, one must have $|\alpha|+|\beta|<1$ (see § 5). The autocovariances are generated by

$$(1-\alpha z_1-\beta z_2)^{-1}(1-\alpha z_1^{-1}-\beta z_2^{-1})^{-1},$$

and yield correlations

$$\rho(s,0)=A^s, \quad \rho(0,t)=B^t,$$

$$\rho(s,t)=\left[\sum_{j=0}^{s}\binom{s+t-j}{t}\alpha^{s-j}\beta^tA^j+\sum_{k=0}^{t}\binom{s+t-k}{s}\alpha^s\beta^{t-k}B^k\right], \tag{58}$$

where

$$\left.\begin{aligned} A&=\frac{1+\alpha^2-\beta^2-\Delta}{2\alpha}, \quad B=\frac{1+\beta^2-\alpha^2-\Delta}{2\beta}, \\ \Delta&=\sqrt{\{(1+\alpha+\beta)(1+\alpha-\beta)(1-\alpha+\beta)(1-\alpha-\beta)\}}. \end{aligned}\right\} \tag{59}$$

Formula (58) holds for $s, t>0$.

The continuous analogue of scheme (56) would be the first-order stochastic differential equation, formally written

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\gamma\right)\xi(x,y)=\epsilon(x,y). \tag{60}$$

Rotating the plane to new axes $X=0$, $Y=0$, inclined at an angle $\tan^{-1}(\beta/\alpha)$ to the old ones, we find the relation may be written

$$\left(\sqrt{(\alpha^2+\beta^2)}\frac{\partial}{\partial X}+\gamma\right)\xi'(X,Y)=\epsilon'(X,Y), \tag{61}$$

where $\xi'(X,Y)=\xi(x,y)$, $\epsilon'(X,Y)=\epsilon(x,y)$. That is, scheme (60) may be regarded as a series of Markoff processes running side by side in the direction $\tan^{-1}(\beta/\alpha)$ and independent one of the other. This indicates the degenerate nature of the first-order scheme.

The simplest second-order scheme (and thus the simplest non-degenerate scheme) is the symmetric autoregression

$$\xi_{st}=\alpha(\xi_{s+1,t}+\xi_{s-1,t}+\xi_{s,t+1}+\xi_{s,t-1})+\epsilon_{st}. \tag{62}$$

Written in the form

$$\left[\Delta_s^2+\Delta_t^2+\left(4-\frac{1}{\alpha}\right)\right]\xi_{st}=\epsilon_{st} \tag{63}$$

(where Δ is the central difference operator) the analogy with the continuous relation, a stochastic Laplace equation, becomes apparent:

$$\left[\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2-\kappa^2\right]\xi(x,y)=\epsilon(x,y). \tag{64}$$

The continuous relation is the easier to handle (as was also the case for relations (56), (60)) due to its essentially less artificial nature. Explicit results have been obtained for scheme (62) (Van der Pol & Bremmer, 1950; Stöhr, 1950), but these are not simple. From the formula (Titchmarsh, 1948, p. 201)

$$\frac{1}{4\pi^2}\iint\frac{e^{i(X\omega_1+Y\omega_2)}\,d\omega_1\,d\omega_2}{(\omega_1^2+\omega_2^2+\kappa^2)^{\mu+1}}=\left(\frac{r}{2\kappa}\right)^{\mu}\frac{K_\mu(\kappa r)}{\Gamma(\mu+1)}\quad(r=\sqrt{(X^2+Y^2)}), \tag{65}$$

we deduce that for scheme (64)

$$\xi(x,y)=\iint_{-\infty}^{+\infty}\epsilon(x+X,y+Y)\,K_0(\kappa r)\,dX\,dY \tag{66}$$

and

$$\phi(X,Y)=\frac{r}{2\kappa}K_1(\kappa r), \tag{67}$$

where K_ν is the modified Bessel function of the second kind, order ν. Since $\lim rK_1(r)=1$ the correlation coefficient corresponding to (67) is

$$\rho(r)=\kappa rK_1(\kappa r). \tag{68}$$

Scheme (64) is a special case of the general second-order stochastic difference equation. According to the usual classification into elliptic, parabolic, and hyperbolic forms, (64) would be regarded as a circular form with centre at the origin. An investigation of the remaining second-order schemes would be of great interest, both theoretically and practically.

The correlation function (68) is of interest in that it may be regarded as the 'elementary' correlation in two dimensions, similar to the exponential $e^{-\alpha|x|}$ in one dimension. Both correlation curves are monotone decreasing, but (68) differs in that it is flat at the origin, and that its rate of decay is slower than exponential.

Attempts have been made to represent two-dimensional correlograms as sums of exponentials, largely because the exponential has proved itself the natural choice in one dimension, and the observed curves display the same monotonic decay. However, it is apparent from the previous paragraphs that the exponential function has no divine right in two dimensions, while the example of the last section indicated that a K_1 function fitted the observations better than did an exponential.

Two-dimensional processes can be constructed which have exponential correlation functions, but these are very artificial. For example, Matérn (1947) has shown that the correlation function $\exp-\alpha\sqrt{(X^2+Y^2)}$ corresponds to a spectral function $(\omega_1^2+\omega_2^2-\alpha^2)^{-\frac{1}{2}}$. The simplest process with such a spectral function may be formally written

$$\left[\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2-\alpha^2\right]^{\frac{1}{2}}\xi(x,y)=\epsilon(x,y) \tag{69}$$

and it is difficult to visualize a physical mechanism which would lead to such a relation.

The examination of the two sets of uniformity data was begun by Mr I. D. Dick of the New Zealand Department of Scientific and Industrial Research, who also in large degree stimulated the author to the present investigation. Thanks are also due to the supervisor of the punched card calculations, Mr E. W. Jones.

REFERENCES

BARTLETT, M. S. (1946). Stochastic processes. Mimeographed N. Carolina lecture notes.

BATCHELOR, L. D. & REED, H. S. (1924). Relation of the variability of yields of fruit trees to the accuracy of field trials. *J. Agric. Res.* **12**, 245–83.

DANIELL, P. J. (1946). Contribution to discussion on stochastic processes. *J. R. Statist. Soc.* B, **8**, 88–90.

DOOB, J. L. (1944). The elementary Gaussian processes. *Ann. Math. Statist.* **15**, 229–82.

MATÉRN, B. (1947). Metoder att uppskatta noggrannheten vid linje- och provytetaxering. *Medd. SkogsforsknInst.* **36**, no. 1.

MERCER, W. B. & HALL, A. D. (1911). The experimental error of field trials. *J. Agric. Sci.* **4**, 107–32.

MORAN, P. A. P. (1950). Notes on continuous stochastic phenomena. *Biometrika*, **37**, 17–23.

NEYMAN, J. (1952). *Lectures and Conferences on Mathematical Statistics and Probability.* Graduate School, U.S. Dep. of Agriculture.

QUENOUILLE, M. H. (1949). Problems in plane sampling. *Ann. Math. Statist.* **20**, 355–75.

STÖHR, A. (1950). Über einige lineare partielle Differenzgleichungen mit konstanten Koeffizienten. *Math. Nachr.* **3**, 295–315.

TITCHMARSH, E. C. (1948). *Introduction to the Theory of Fourier Integrals.* Oxford University Press.

VAN DER POL, B. & BREMMER, H. (1950). *Operational Calculus.* Cambridge University Press.

WHITTLE, P. (1952). Tests of fit in time series. *Biometrika*, **39**, 309–18.

WHITTLE, P. (1953). The analysis of multiple stationary time series. *J. R. Statist. Soc*, B, **15**, 125–39.

WIENER, N. (1949). *The Extrapolation, Interpolation and Smoothing of Stationary Time-Series.* New York. Wiley.

WILLIAMS, R. M. (1954). The choice of sampling intervals for systematic samples from populations with stationary correlation. (Unpublished paper.)

WOLD, H. (1938). *A Study in the Analysis of Stationary Time Series.* Upsala.

[EDITORIAL NOTE. The preceding paper by Whittle (on p. 443) and the following paper by Patankar (on pp. 459–60) both use as an illustration of theory Mercer & Hall's (1911) wheat-plot data. On a first inspection it might be thought that the serial correlations in (i) the column $s = 0$ and (ii) in the row $t = 0$ of Whittle's Table 1, p. 443, should correspond with Patankar's correlations described as (i) 'along the rows, results over the whole data, original' (last column of Table 5·2, p. 460) and (ii) 'along the columns, results over the whole data' (last column of Table 5·1). Reference to the authors has, however, confirmed that the quantities computed are differently defined. Thus if

$i = 1, 2, \ldots, 25$ denotes columns running from west to east,

$j = 1, 2, \ldots, 20$ denotes rows running from north to south,

the co-variance in Whittle's expression for r_{s0} is

$$\sum_{i=1}^{25}\sum_{j=1}^{20-s} x_{i,j}x_{i,j+s} - \sum_{i=1}^{25}\sum_{j=1}^{20-s} x_{i,j} \times \sum_{i=1}^{25}\sum_{j=1}^{20-s} x_{i,j+s}/\{25(20-s)\},$$

with similar expressions for the variances. The corresponding covariance used by Patankar is, however,

$$\sum_{i=1}^{25}\left\{\sum_{j=1}^{20-s} x_{i,j}x_{i,j+s} - \sum_{j=1}^{20-s} x_{i,j} \times \sum_{j=1}^{20-s} x_{i,j+s}/(20-s)\right\},$$

with similar expressions for the variances. Thus Whittle bases his serial correlations on the total variation and covariation while Patankar's correlations are based on within-column (or within-row) variation and covariation. As might be expected, Whittle's correlations are larger than Patankar's because of the variation between rows and columns.]

THE GOODNESS OF FIT OF FREQUENCY DISTRIBUTIONS OBTAINED FROM STOCHASTIC PROCESSES

By V. N. PATANKAR

The Ahmedabad Textile Industry's Research Association, India

1. Introduction

The classical χ^2 test of goodness of fit introduced by K. Pearson (1900) is concerned with the agreement between the distribution of a set of sample values and a theoretical distribution. The data are usually divided into r mutually exclusive groups and the corresponding group frequencies will be denoted by $n_1, n_2, \ldots, n_r$. If the group frequencies expected from the parent distribution be $m_1, m_2, \ldots, m_r$, Pearson's measure of deviation is simply

$$\chi^2 = \sum_{i=1}^{r} \frac{(n_i - m_i)^2}{m_i}. \tag{1}$$

Then the test is provided by the fact that as the sample size tends to infinity the sampling distribution of χ^2 tends to that of the standard χ^2 distribution with $r-1$ degrees of freedom. The importance of this test is realized by noting that the χ^2 criterion can be shown to be equivalent to the likelihood-ratio test in large samples.

The fundamental assumption underlying the above test is the independent and unordered nature of the sample observations. It is true that in many cases encountered in practice this assumption is satisfied, and hence this test has been widely used. However, with the recent developments in the theory of stochastic processes, some interesting cases have been found where there is some dependence among the sample observations (which may not be wholly attributed to random fluctuations).

The first attempt to consider the effect of dependence among stochastic process data on the χ^2 test was made by Bartlett (1951). He has discussed the problem of testing the goodness of fit of a theoretical model to a 'finitely dependent' probability chain, by which is meant a stochastic process defined for discrete values of both variable and parameter, whose probability dependence does not extend more than a finite number of intervals. Bartlett, while primarily concerned with a complete test of the stochastic process model, showed that under certain assumptions (which he has stated in his paper) the asymptotic distribution of the marginal frequencies n_i is normal, but (1) has no longer a standard χ^2 distribution (see also Good, 1953).

Taking account of the fact that the marginal frequencies n_i $(i = 1, \ldots, r-1)$ have asymptotically the multivariate normal distribution, with variance-covariance matrix $\mathbf{V} = (\sigma_{ij})$, say, P. S. Neall (1952)* has suggested the following likelihood-ratio criterion to test the goodness of fit; he takes

$$\lambda = (n_1 - m_1, \ldots, n_{r-1} - m_{r-1})\, \mathbf{V}^{-1} \begin{pmatrix} n_1 - m_1 \\ \vdots \\ n_{r-1} - m_{r-1} \end{pmatrix} \tag{2}$$

* I am indebted to Prof. M. S. Bartlett for having informed me about the unpublished work of Mr P. S. Neall.

to have a χ^2 distribution with $r-1$ degrees of freedom. In the independence case $\mathbf{V}^{-1}$ is simply

$$\left(\frac{\delta_{ij}}{m_i}+\frac{1}{m_r}\right),$$

where δ_{ij} is the Kronecker delta, and hence λ reduces to the expression (1). In the case of dependence $\mathbf{V}^{-1}$ does not reduce to the above form and $\mathbf{V}^{-1}$ has to be calculated separately for each distribution. There are two serious practical disadvantages in Neall's test. One is, if we assume that $\mathbf{V}$ is known exactly for a certain distribution, to find the inverse of $\mathbf{V}$ may be a practically impossible task for a large number of classes. Moreover, even in the simple cases of the Poisson and normal Markoff processes, the calculations of $\sigma_i^2 = \sigma_{ii}$ and σ_{ij} to the exact degree of approximation may be difficult.

Bearing this in mind, we may turn back to the standard criterion (1) which is comparatively easy to calculate and would still from its functional form indicate any departures from the null hypothesis that the given sample has been drawn from a specified parent distribution. On the assumption of the normality of the frequencies n_i, we can find the mean and variance of (1), and try to fit an approximate χ^2 distribution by the method of moments. This is the object of the present paper.

2. Modified χ^2 test of goodness of fit

If $n_i - m_i = x_i$ $(i = 1, \ldots, r-1)$, then under certain assumptions (see Bartlett, 1951), the joint distribution of the x_i's tends to a multivariate normal distribution, i.e. the probability density

$$P\{\mathbf{x}\} \to \frac{1}{(2\pi)^{\frac{1}{2}(r-1)} |\, V\,|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}\},$$

where $\mathbf{x}$ is the column vector $(x_1, \ldots, x_{r-1})$ and (1) reduces to the form

$$\chi^2 = \mathbf{x}'\mathbf{Q}\mathbf{x},$$

$\mathbf{Q}$ being $\left(\frac{\delta_{ij}}{m_i}+\frac{1}{m_r}\right)$.

The characteristic function of χ^2 is

$$\phi_{\chi^2}(t) = \frac{1}{|\, \mathbf{I} - 2it\mathbf{V}\mathbf{Q}\,|^{\frac{1}{2}}} = \frac{1}{\prod_{j=1}^{r-1}(1-2it\theta_j)^{\frac{1}{2}}},$$

where the θ_j's are the latent roots of the matrix $\mathbf{VQ}$. Hence the sth cumulant of χ^2 is

$$\kappa_s = 2^{s-1}(s-1)! \sum_{j=1}^{r-1} \theta_j^s.$$

If $\mathbf{VQ} = (a_{ij})$, then $\sum_{i=1}^{r-1} \theta_i = \sum_{i=1}^{r-1} a_{ii}$, $\sum_{i=1}^{r-1} \theta_i^2 = \sum_{i,j=1}^{r-1} a_{ij}a_{ji}$,

and hence we obtain the expressions

$$\left.\begin{aligned} E(\chi^2) &= \sum_{i=1}^{r} \frac{\sigma_i^2}{m_i} = A \quad \text{say}, \\ V(\chi^2) &= 2\sum_{i,j=1}^{r} \frac{\sigma_{ij}^2}{m_i m_j} = 2B \quad \text{say}. \end{aligned}\right\} \tag{3}$$

Then a better approximation to the standard χ^2 is obtained by taking $A\chi^2/B$ to have an asymptotic χ^2 distribution with A^2/B degrees of freedom. This we call the modified χ^2 test of goodness of fit.

In the case of independence,

$$\sigma_i^2 = m_i - \frac{m_i^2}{n}, \quad \sigma_{ij} = -\frac{m_i m_j}{n},$$

where n is the sample size. These results do not hold good for the dependence case. To obtain these expressions in the latter case, we employ a familiar device of putting

$$X_{(t)i} = \begin{cases} 1 & \text{if at any time } t, \text{ the system is in the } i\text{th group } (i = 1, \ldots, r;\ t = 1, \ldots, n), \\ 0 & \text{otherwise.} \end{cases}$$

Then $n_i = \sum_{t=1}^{n} X_{(t)i}$ and after some algebra, we get

$$\left.\begin{aligned} \sigma_i^2 &= m_i - m_i^2 + 2\sum_{s=1}^{n-1}(n-s)P_{ii}^{(s)}, \\ \sigma_{ij} &= -m_i m_j + \sum_{s=1}^{n-1}(n-s)P_{ij}^{(s)} + \sum_{s=1}^{n-1}(n-s)P_{ji}^{(s)}, \end{aligned}\right\} \tag{4}$$

where $P_{ij}^{(s)}$ is the absolute probability of obtaining the observations belonging to the ith and jth groups at times t and $t+s$, respectively, for all t (assuming stationarity).

3. Poisson-Markoff process

The Poisson-Markoff process* as defined below is playing quite an important role in the theory of stochastic processes, and in many interesting situations the observations are found to be governed by this process. One such example is given by Chandrasekhar (1943), who has fitted a Poisson distribution to a certain sequence of observations by Westgren in his classical experiment on colloidal particles, to confirm the theory developed by Smoluchowsky. The various observations in the sequence are not independent, and hence the validity of the classical theory to test the goodness of fit may be doubted. It is possible to find the approximate values of the mean and variance of χ^2 and test the goodness of fit by the theory developed above.

From the formulae (333) and (364) in Chandrasekhar's paper mentioned above, it is possible to give the expressions for p_i (the probability of the observation being in the ith group) and $P_{ij}^{(s)}$ as follows:

$$\left.\begin{aligned} p_i &= e^{-\nu}\frac{\nu^i}{i!} \quad (i = 0, 1, 2, \ldots), \\ P_{ij}^{(s)} &= e^{-2\nu+\nu\rho_1^s}\sum_{t=0}^{\min(i,j)}\frac{\nu^{i+j-t}(1-\rho_1^s)^{i+j-2t}\rho_1^{st}}{(i-t)!\,(j-t)!\,t!} \quad (i, j = 0, 1, 2, \ldots), \end{aligned}\right\} \tag{5}$$

where ν is the mean value and ρ_1 the correlation coefficient between the two consecutive observations.

Suppose now that it is required to test the hypothesis that a given sample of n values $x_1, x_2, \ldots, x_n$ is drawn from some Poisson-Markoff distribution with a known value of the

* This should not be confused with the simple process with random unit jumps at rate λ, sometimes referred to as the 'Poisson process'.

parameter ν. Every x_μ is equal to some non-negative integer i, and we arrange the x_μ according to their values into $k+1$ groups, pooling the data for the largest value of i, where the observations are few. Suppose we obtain in this way

n_i observations with $x = i$, where $i = 0, 1, \ldots, k-1$,

n'_k observations with $x \geqslant k$.

The corresponding expected frequencies in the various groups are

$$m_i = np_i \quad (i = 0, 1, \ldots, k-1),$$

$$m'_k = np'_k = n \sum_{i \geqslant k} p_i.$$

The joint probabilities are

$$P^{(s)}_{ij} \quad (i, j = 0, 1, \ldots, k-1),$$

$$P'^{(s)}_{ik} = \sum_{j=k}^{\infty} P^{(s)}_{ij} \quad (i = 0, 1, \ldots, k-1),$$

$$P'^{(s)}_{kk} = \sum_{i,j=k}^{\infty} P^{(s)}_{ij}.$$

Substituting the values of the above expressions from (5) into (4) and expanding the functions under the summation by Taylor's theorem in powers of ρ_1, we get

$$\sigma_i^2 = m_i - \frac{m_i^2}{n} + 2\frac{n-1}{n^2}\frac{A_i^2 m_i^2}{\nu}\rho_1 + \left\{\frac{n-1}{n^2}\frac{B_i^2 m_i^2}{\nu^2} + 2\frac{n-2}{n^2}\frac{A_i^2 m_i^2}{\nu}\right\}\rho_1^2 + \left\{\frac{n-1}{n^2}\frac{C_i^2 m_i^2}{3\nu^3} + 2\frac{n-3}{n^2}\frac{A_i^2 m_i^2}{\nu}\right\}\rho_1^3 + O(n\rho_1^4) \quad (i = 0, 1, 2, \ldots, k-1),$$

$$\sigma_k'^2 = m'_k - \frac{m_k'^2}{n} + 2\frac{n-1}{n^2}\frac{A_k'^2 m_k'^2}{\nu}\rho_1 + \left\{\frac{n-1}{n^2}\frac{B_k'^2 m_k'^2}{\nu^2} + 2\frac{n-2}{n^2}\frac{A_k'^2 m_k'^2}{\nu}\right\}\rho_1^2 + \left\{\frac{n-1}{n^2}\frac{C_k'^2 m_k'^2}{3\nu^3} + 2\frac{n-3}{n^2}\frac{A_k'^2 m_k'^2}{\nu}\right\}\rho_1^3 + O(n\rho_1^4),$$

$$\sigma_{ij} = -\frac{m_i m_j}{n} + 2\frac{n-1}{n^2}\frac{A_i A_j m_i m_j}{\nu}\rho_1 + \left\{\frac{n-1}{n^2}\frac{B_i B_j m_i m_j}{\nu^2} + 2\frac{n-2}{n^2}\frac{A_i A_j m_i m_j}{\nu}\right\}\rho_1^2 + \left\{\frac{n-1}{n^2}\frac{C_i C_i m_i m_j}{3\nu^3} + 2\frac{n-3}{n^2}\frac{A_i A_j m_i m_j}{\nu}\right\}\rho_1^3 + O(n\rho_1^4) \quad (i, j = 0, 1, \ldots, k-1),$$

$$\sigma'_{ik} = -\frac{m_i m'_k}{n} + 2\frac{n-1}{n^2}\frac{A_i A'_k m_i m'_k}{\nu}\rho_1 + \left\{\frac{n-1}{n^2}\frac{B_i B'_k m_i m'_k}{\nu^2} + 2\frac{n-2}{n^2}\frac{A_i A'_k m_i m'_k}{\nu}\right\}\rho_1^2 + \left\{\frac{n-1}{n^2}\frac{C_i C'_k m_i m'_k}{3\nu^3} + 2\frac{n-3}{n^2}\frac{A_i A'_k m_i m'_k}{\nu}\right\}\rho_1^3 + O(n\rho_1^4) \quad (i = 0, 1, \ldots, k-1),$$

where

$$A_i = \nu - i,$$

$$B_i = \nu^2 - 2\nu i + i(i-1),$$

$$C_i = \nu^3 - 3\nu^2 i + 3\nu i(i-1) - i(i-1)(i-2),$$

$$A'_k = \nu - \sum_{i=k}^{\infty} \frac{i m_i}{m'_k},$$

$$B'_k = \nu^2 - 2\nu \sum_{i=k}^{\infty} \frac{i m_i}{m'_k} + \sum_{i=k}^{\infty} i(i-1)\frac{m_i}{m'_k},$$

$$C'_k = \nu^3 - 3\nu^2 \sum_{i=k}^{\infty} \frac{i m_i}{m'_k} + 3\nu \sum_{i=k}^{\infty} i(i-1)\frac{m_i}{m'_k} - \sum_{i=k}^{\infty} i(i-1)(i-2)\frac{m_i}{m'_k}.$$

We have
$$\sum_{i=0}^{k-1} A_i m_i + A'_k m'_k = 0,$$
$$\sum_{i=0}^{k-1} B_i m_i + B'_k m'_k = 0,$$
$$\sum_{i=0}^{k-1} C_i m_i + C'_k m'_k = 0,$$

and we write approximately, for large k,
$$\sum_{i=0}^{k-1} A_i^2 m_i + A_k'^2 m'_k \sim \sum_{i=0}^{\infty} A_i^2 m_i = n\nu,$$
$$\sum_{i=0}^{k-1} B_i^2 m_i + B_k'^2 m'_k \sim \sum_{i=0}^{\infty} B_i^2 m_i = 2n\nu^2,$$
$$\sum_{i=0}^{k-1} C_i^2 m_i + C_k'^2 m'_k \sim \sum_{i=0}^{\infty} C_i^2 m_i = 6n\nu^3,$$
$$\sum_{i=0}^{k-1} A_i B_i m_i + A'_k B'_k m'_k \sim \sum_{i=0}^{\infty} A_i B_i m_i = 0.$$

With these values, expressions (3) reduce, for large n, to
$$E(\chi^2) \sim k + 2\rho_1 + 4\rho_1^2 + 4\rho_1^3 + O(\rho_1^4), \tag{6a}$$
$$V(\chi^2) \sim 2k + 8\rho_1 + 24\rho_1^2 + 32\rho_1^3 + O(\rho_1^4). \tag{6b}$$

We can rely on these formulae only for small ρ_1, when the fourth powers of ρ_1 are small compared with the third power.

The higher degree terms in ρ_1 can be calculated, but they require laborious algebra.

To illustrate these results, Chandrasekhar's fitting of the Poisson-Markoff distribution may be considered. The 1583 observations are grouped into seven classes giving χ^2 equal to 8·9501. The mean value ν is known to be 1·428 and the correlation coefficient calculated from the theoretical formula is 0·606.

From (6) the mean is 9·571124 (the contribution due to the term in ρ_1^3 in this being 0·890180); and the variance is 32·783104 (the contribution due to the term in ρ_1^3 in this being 7·121440). As noted above, ρ_1 seems large enough to doubt the accuracy of the variance and hence the calculation of the higher degree terms seems essential. The higher degree terms for the Poisson process are not yet found due to the heavy algebra involved, but for the normal Markoff process (which is discussed in § 4 later), all the further terms are easily calculated (see (8)). Since the terms up to the third degree in (8) coincide with those in (6) we have taken (8) as an approximation to (6), and have used it for the Poisson-Markoff process as well. Then we have the mean approximately equal to 11·6298, while the variance is approximately 57·91344. The modified χ^2 is equal to 3·5946 with 4·67 degrees of freedom, which is quite insignificant.

4. Normal Markoff process

Let there be a sample of n values, $x_1, x_2, \ldots, x_n$ and the probability for such a sample be given by the normal Markoff distribution, viz.

$$p(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)}\,\sigma} e^{-(x_1-m)^2/2\sigma^2} \prod_{r=2}^{n} \frac{1}{\sqrt{(2\pi)}\,\sigma\sqrt{(1-\rho_1^2)}} \exp\left\{-\frac{[(x_r-m)-\rho_1(x_{r-1}-m)]^2}{2\sigma^2(1-\rho_1^2)}\right\}. \tag{7}$$

The sample values are grouped into k classes, the ith class containing n_i observations situated in the interval $(\xi_i - \frac{1}{2}h, \xi_i + \frac{1}{2}h)$, where $\xi_i = \xi_1 + (i-1)h$. The case of finite classes (cf. Bartlett, 1951) is artificially obtained by taking the first and the last classes as $(-\infty, \xi_1 + \frac{1}{2}h)$ and $(\xi_k - \frac{1}{2}h, \infty)$ in the same way as is done in the case of independent observations. We want to test the hypothesis that the sample has been drawn from some normal Markoff population with the specified values of m and σ. If this hypothesis is true the probability p_i corresponding to the ith class is

$$p_i = \frac{1}{\sqrt{(2\pi)}\,\sigma} \int e^{-(x-m)^2/2\sigma^2}\, dx,$$

where the integral is extended over the ith class interval. Then $np_i = m_i$ is the expected frequency in the ith class. Also $P_{ij}^{(s)}$ as defined in (4) is given by

$$P_{ij}^{(s)} = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho_1^{2s})}} \int_i \int_j \exp\left\{ -\frac{1}{2\sigma^2(1-\rho_1^{2s})} [(x-m)^2 - 2\rho_1^s (x-m)(y-m) + (y-m)^2] \right\} dx\, dy,$$

Hence, from (4),

$$\frac{\sigma_i^2}{m_i} = 1 - m_i + 2 \sum_{s=1}^{n-1} \frac{n-s}{n} \frac{1}{\sqrt{(2\pi)}\,\sigma\sqrt{(1-\rho_1^{2s})}} \times \frac{\int_i \int_i \exp\left\{ -\frac{1}{2\sigma^2(1-\rho_1^{2s})} [(x-m)^2 - 2\rho_1^s (x-m)(y-m) + (y-m)^2] \right\} dx\, dy}{\int_i e^{-(x-m)^2/2\sigma^2}\, dx}.$$

For small h, an approximate solution may be obtained simply by replacing the functions under the integrals by their values at the midpoint ξ_i of the corresponding class interval, multiplied by the area of integration.* Then

$$\frac{\sigma_i^2}{m_i} = 1 - m_i + 2 \sum_{s=1}^{n-1} \frac{n-s}{n} \frac{1}{1-\rho_1^s} \frac{h}{\sqrt{(2\pi)}\,\sigma'} e^{-(\xi_i - m)^2/2\sigma'^2},$$

where $\sigma'^2 = \sigma^2 \dfrac{1+\rho_1^s}{1-\rho_1^s}$. Hence

$$\sum_{i=1}^{k} \frac{\sigma_i^2}{m_i} = k - n + 2 \sum_{s=1}^{n-1} \frac{n-s}{n} \frac{1}{1-\rho_1^s},$$

since

$$\sum_{i=1}^{k} \frac{h}{\sqrt{(2\pi)}\,\sigma'} e^{-(\xi_i - m)^2/2\sigma'^2}$$

is equal to the sum of the ordinates of a normal curve (with S.D. σ') multiplied by the length of the class interval, and becomes approximately equal to one. Hence

$$E(\chi^2) \sim k - 1 + 2 \sum_{s=1}^{n-1} \frac{n-s}{n} \frac{\rho_1^s}{1-\rho_1^s},$$

which for large n, can be written as

$$E(\chi^2) \sim k - 1 + 2 \sum_{s=1}^{n-1} \frac{\rho_1^s}{1-\rho_1^s}. \tag{8a}$$

* I wish to acknowledge my thanks to Dr B. L. Welch for this helpful suggestion.

Employing exactly a similar argument for $\sigma_{ij}/\sqrt{(m_i m_j)}$ and putting

$$\frac{h^2}{2\pi\sigma^2\sqrt{(1-\rho_1^2)}}\sum_{i,j=1}^{k}\exp\left\{-\frac{1}{2\sigma^2(1-\rho_1^2)}[(\xi_i-m)^2-2\rho_1(\xi_i-m)(\xi_j-m)+(\xi_j-m)^2]\right\}\simeq 1$$

we get
$$V(\chi^2)\sim 2(k-1)+8\sum_{s=1}^{n-1}\frac{n-s}{n}\frac{\rho_1^s}{1-\rho_1^2}+8\sum_{s,t=1}^{n-1}\frac{n-s}{n}\frac{n-t}{n}\frac{\rho_1^{s+t}}{1-\rho_1^{s+t}},$$

which again, for large n, becomes

$$V(\chi^2)\sim 2(k-1)+8\sum_{s=1}^{n-1}\frac{\rho_1^s}{1-\rho_1^s}+8\sum_{s,t=1}^{n-1}\frac{\rho_1^{s+t}}{1-\rho_1^{s+t}}\sim 2(k-1)+8\sum_{s=1}^{n-1}\frac{\rho_1^s}{(1-\rho_1^s)^2}. \tag{8b}$$

To illustrate the above theory we consider the artificial series (nos. 7, 9, 11, 13 and 15) of M. G. Kendall (1949). There he takes the series generated by the equation

$$u_{t+1}-\rho_1 u_t=\epsilon_{t+1},$$

where ϵ_{t+1} is a random normal element (0, 1). Then u_t has a normal Markoff distribution with mean zero and variance $1/(1-\rho_1^2)$ and the correlation coefficient ρ_1. A normal curve was fitted to each of the series with the same number of classes, viz. 10, in each case.

Table 4·1 gives the modified χ^2 test of goodness of fit for the above series.

Table 4·1. *Modified χ^2 test of goodness of fit for M. G. Kendall's series*

Series no.	ρ_1	Observed χ^2	$E(\chi^2)=A$	$V(\chi^2)=2B$	Modified χ^2 $A\chi^2/B$	D.F. A^2/B
15	0·1	6·579	9·244	19·080	6·3749	8·9572
13	0·3	8·397	10·134	24·088	7·0654	8·5269
11	0·5	21·746	12·214	39·952	13·2962	7·4681
9	0·7	17·231	18·512	110·560	5·7703	6·1992
7	0·9	26·833	63·172	1165·816	2·9080	6·8462

All the values are insignificant, and as far as one can rely on this test, there is no reason to doubt the fit.

In general for any stationary normal process, we obtain

$$E(\chi^2)\sim k-1+2\sum_{s=1}^{n-1}\frac{\rho_s}{1-\rho_s},\quad V(\chi^2)\sim 2(k-1)+8\sum_{s=1}^{n-1}\frac{\rho_s}{1-\rho_s}+8\sum_{s,t=1}^{n-1}\frac{\rho_s\rho_t}{1-\rho_s\rho_t}. \tag{9}$$

5. Two-dimensional normal Markoff process

Let the N observations be denoted by x_{ij} $(i=1,2,\ldots,n_1;\ j=1,2,\ldots,n_2)$, where $N=n_1n_2$ (with n_1 columns and n_2 rows) and x_{ij} are supposed to be drawn from a normal population with

$$E(x_{ij})=m_{ij},\quad V(x_{ij})=\sigma_{ij}^2,$$

$$E(x_{ij}-m_{ij})(x_{i+s,j+t}-m_{i+s,j+t})=\rho_{ijst}\sigma_{ij}\sigma_{i+s,j+t}.$$

This will be a stationary process, if

$$m_{ij}=m,\quad \sigma_{ij}^2=\sigma^2,\quad \rho_{ijst}=\rho_{st}\quad\text{and}\quad \rho_{st}=\rho_{-s,-t}=\rho_{-s,t}=\rho_{s,-t},$$

and we suppose that these conditions are satisfied. Then it is quite easy to note that the marginal distributions of x_{ij} and $(x_{ij}, x_{i+s,j+t})$ are given by

$$p(x_{ij}) = \frac{1}{\sqrt{(2\pi)}\,\sigma} e^{-(x_{ij}-m)^2/2\sigma^2},$$

$$p(x_{ij}, x_{i+s,j+t}) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho_{st}^2)}} \times \exp\left\{\frac{-1}{2\sigma^2(1-\rho_{st}^2)}[(x_{ij}-m)^2 - 2\rho_{st}(x_{ij}-m)(x_{i+s,j+t}-m) + (x_{i+s,j+t}-m)^2]\right\}.$$

Let the sample of N values x_{ij} be grouped into k classes, the ith class containing n_i observations situated in the interval $(\xi_i - \frac{1}{2}h, \xi_i + \frac{1}{2}h)$, where $\xi_i = \xi_1 + (i-1)h$. We take the first and the last classes to be $(-\infty, \xi_1 + \frac{1}{2}h)$ and $(\xi_k - \frac{1}{2}h, \infty)$ as usual. We want to test the hypothesis that the sample has been drawn from some normal stationary population with the specified values of m and σ. If the hypothesis is true, the probability p_i corresponding to the ith class is

$$p_i = \frac{1}{\sqrt{(2\pi)}\,\sigma}\int_i e^{-(x-m)^2/2\sigma^2}\,dx,$$

where the integral is extended over the ith class interval. Then $Np_i = m_i$ is the corresponding expected frequency. To obtain the variance and covariance of the marginal frequencies we employ the same device as in the case of one-dimensional processes by putting

$$X_{(p,q)i} = \begin{cases} 1 & \text{if at any time } (p,q) \text{ the observation belongs to the } i\text{th class} \\ & \quad (p = 1, \ldots, n_1;\ q = 1, \ldots, n_2;\ i = 1, \ldots, k), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{p,q=1}^{n_1,n_2} X_{(p,q)i} = n_i$; and after some algebra,

$$\sigma_i^2 = m_i - m_i^2 + 2\sum_{s=1}^{n_1-1} n_2(n_1-s)\,P_{ii}^{(s,0)} + 2\sum_{t=1}^{n_2-1} n_1(n_2-t)\,P_{ii}^{(0,t)} + 2\sum_{s=1}^{n_1-1}\sum_{\substack{t=-n_2+1 \\ t\neq 0}}^{n_2-1} (n_1-s)(n_2-|t|)\,P_{ii}^{(s,t)}$$

and

$$\sigma_{ij} = -m_i m_j + \sum_{s=1}^{n_1-1} n_2(n_1-s)\,P_{ij}^{(s,0)} + \sum_{s=1}^{n_1-1} n_2(n_1-s)\,P_{ji}^{(s,0)} + \sum_{t=1}^{n_2-1} n_1(n_2-t)\,P_{ij}^{(0,t)} + \sum_{t=1}^{n_2-1} n_1(n_2-t)\,P_{ji}^{(0,t)} + \sum_{s=1}^{n_1-1}\sum_{\substack{t=-n_2+1 \\ t\neq 0}}^{n_2-1} (n_1-s)(n_2-|t|)\,P_{ij}^{(s,t)} + \sum_{s=1}^{n_1-1}\sum_{\substack{t=-n_2+1 \\ t\neq 0}}^{n_2-1} (n_1-s)(n_2-|t|)\,P_{ji}^{(s,t)},$$

where $P_{ij}^{(s,t)}$ is the absolute probability of obtaining the observations belonging to the ith and jth classes at times (p,q) and $(p+s, q+t)$ respectively, for all p, q (assuming stationarity).

We get, proceeding on the same lines as for the one-dimensional process,

$$E(\chi^2) \sim k - 1 + 2\sum_{s=1}^{n_1-1}\frac{n_1-s}{n_1}\frac{\rho_{s0}}{1-\rho_{s0}} + 2\sum_{t=1}^{n_2-1}\frac{n_2-t}{n_2}\frac{\rho_{0t}}{1-\rho_{0t}} + 4\sum_{s,t=1}^{n_1-1,\,n_2-1}\frac{n_1-s}{n_1}\frac{n_2-t}{n_2}\frac{\rho_{st}}{1-\rho_{st}},$$

which for large n_1 and n_2,

$$\sim k - 1 + 2\sum_{s=1}^{n_1-1}\frac{\rho_{s0}}{1-\rho_{s0}} + 2\sum_{t=1}^{n_2-1}\frac{\rho_{0t}}{1-\rho_{0t}} + 4\sum_{s,t=1}^{n_1-1,\,n_2-1}\frac{\rho_{st}}{1-\rho_{st}}, \qquad (10a)$$

and

$$V(\chi^2)\sim 2(k-1)+8\sum_{s=1}^{n_1-1}\frac{n_1-s}{n_1}\frac{\rho_{s0}}{1-\rho_{s0}}+8\sum_{t=1}^{n_2-1}\frac{n_2-t}{n_2}\frac{\rho_{0t}}{1-\rho_{0t}}$$
$$+16\sum_{s=1}^{n_1-1}\sum_{t=1}^{n_2-1}\frac{n_1-s}{n_1}\frac{n_2-t}{n_2}\frac{\rho_{st}}{1-\rho_{st}}+8\sum_{s=1}^{n_1-1}\sum_{s'=1}^{n_1-1}\frac{n_1-s}{n_1}\frac{n_1-s'}{n_1}\frac{\rho_{s0}\rho_{s'0}}{1-\rho_{s0}\rho_{s'0}}$$
$$+8\sum_{t=1}^{n_2-1}\sum_{t'=1}^{n_2-1}\frac{n_2-t}{n_2}\frac{n_2-t'}{n_2}\frac{\rho_{0t}\rho_{0t'}}{1-\rho_{0t}\rho_{0t'}}$$
$$+32\sum_{s=1}^{n_1-1}\sum_{s'=1}^{n_1-1}\sum_{t=1}^{n_2-1}\sum_{t'=1}^{n_2-1}\frac{n_1-s}{n_1}\frac{n_1-s'}{n_1}\frac{n_2-t}{n_2}\frac{n_2-t'}{n_2}\frac{\rho_{st}\rho_{s't'}}{1-\rho_{st}\rho_{s't'}}$$
$$+16\sum_{s=1}^{n_1-1}\sum_{t=1}^{n_2-1}\frac{n_1-s}{n_1}\frac{n_2-t}{n_2}\frac{\rho_{s0}\rho_{0t}}{1-\rho_{s0}\rho_{0t}}$$
$$+32\sum_{s=1}^{n_1-1}\sum_{s'=1}^{n_1-1}\sum_{t=1}^{n_2-1}\frac{n_1-s}{n_1}\frac{n_1-s'}{n_1}\frac{n_2-t}{n_2}\frac{\rho_{st}\rho_{s'0}}{1-\rho_{st}\rho_{s'0}}$$
$$+32\sum_{s=1}^{n_1-1}\sum_{t=1}^{n_2-1}\sum_{t'=1}^{n_2-1}\frac{n_1-s}{n_1}\frac{n_2-t}{n_2}\frac{n_2-t'}{n_2}\frac{\rho_{st}\rho_{0t'}}{1-\rho_{st}\rho_{0t'}},$$

which again, for large n_1 and n_2 becomes

$$V(\chi^2)\sim 2(k-1)+8\sum_{s=1}^{n_1-1}\frac{\rho_{s0}}{1-\rho_{s0}}+8\sum_{t=1}^{n_2-1}\frac{\rho_{0t}}{1-\rho_{0t}}$$
$$+16\sum_{s,t=1}^{n_1-1,n_2-1}\frac{\rho_{st}}{1-\rho_{st}}+8\sum_{s,s'=1}^{n_1-1}\frac{\rho_{s0}\rho_{s'0}}{1-\rho_{s0}\rho_{s'0}}$$
$$+8\sum_{t,t'=1}^{n_2-1}\frac{\rho_{0t}\rho_{0t'}}{1-\rho_{0t}\rho_{0t'}}+32\sum_{s,s'=1}^{n_1-1}\sum_{t,t'=1}^{n_2-1}\frac{\rho_{st}\rho_{s't'}}{1-\rho_{st}\rho_{s't'}}$$
$$+16\sum_{s=1}^{n_1-1}\sum_{t=1}^{n_2-1}\frac{\rho_{s0}\rho_{0t}}{1-\rho_{s0}\rho_{0t}}+32\sum_{s,s'=1}^{n_1-1}\sum_{t=1}^{n_2-1}\frac{\rho_{st}\rho_{s'0}}{1-\rho_{st}\rho_{s'0}}$$
$$+32\sum_{s=1}^{n_1-1}\sum_{t,t'=1}^{n_2-1}\frac{\rho_{st}\rho_{0t'}}{1-\rho_{st}\rho_{0t'}}. \qquad (10b)$$

There are difficulties in defining a general Markoff process in two dimensions, but analogous to the one-dimensional linear process result $\rho_s=\rho_1^{|s|}$, Quenouille (1949) has defined a process in two dimensions with different forms of the correlation function ρ_{st}. We consider only the form $\rho_{st}=\rho_1^{|s|}.\rho_2^{|t|}$, which he calls a degenerate Markoff process of the first order. With proper substitutions in (10), we get the mean and variance of χ^2 for the normal Markoff process with $\rho_{st}=\rho_1^{|s|}\rho_2^{|t|}$:

$$E(\chi^2)\sim k-1+2\sum_r\frac{\rho_1^r+\rho_2^r}{(1-\rho_1^r)(1-\rho_2^r)}, \qquad (11a)$$

$$V(\chi^2)\sim 2(k-1)+8\sum_r\frac{(\rho_1^r+\rho_2^r)(1+\rho_1^r\rho_2^r)}{(1-\rho_1^r)^2(1-\rho_2^r)^2}. \qquad (11b)$$

To illustrate the above theory we fit a normal curve to the wheat-crop data for yield of grain given by Mercer & Hall (1911) and test the goodness of fit by the modified χ^2 method (but not by the classical χ^2 method, since it is natural to expect some correlation in the yields of plots adjacent to each other). There are 500 plots in a rectangular field arranged in 20 rows and 25 columns. The data can be regarded as homogeneous as far as the variations

from north to south are concerned, but there is a trend in the data from west to east, which can be approximately represented by

$$Y = 3{\cdot}9485 - 0{\cdot}019041(X - 13),$$

where Y represents the expected mean yield in each column and X is an auxiliary variable taking values 1–25 corresponding to the columns from west to east. To make the data homogeneous, it is necessary to eliminate this trend. To keep the same mean for this modified data as that of the original data, we subtract the quantity $-0{\cdot}019041(X-13)$ from each of the observed values of the data. This we shall call the modified wheat-crop data. Next the serial correlations for the rows and columns of the modified data were calculated. These values are given in Tables 5·1 and 5·2. The last two columns in Table 5·2 give the serial correlations over the whole data, first column giving for the modified wheat crop data, and the second for the original data (with this trend in it). For Table 5·1, the correlations are

Table 5·1. *Serial correlations along the columns*

	1	2	3	4	5	6	7
r_1	0·2747	0·6766	−0·0677	0·0181	0·0202	0·3446	0·2495
r_2	·1454	·2454	− ·1201	− ·1379	− ·2612	·0953	·3408
r_3	− ·0050	− ·0008	− ·3252	·0105	·1542	− ·1544	·2823
r_4	·1492	− ·1619	·3058	− ·2037	·3259	− ·4245	·0555
r_5	− ·0415	− ·2270	·1400	− ·1772	·3726	− ·3626	·0645
	8	9	10	11	12	13	14
r_1	0·3875	0·4614	0·0157	0·3896	0·3318	0·0720	0·3697
r_2	·1358	− ·0587	·0087	·0153	− ·0851	− ·1647	·4342
r_3	− ·0212	− ·3811	·1325	− ·3479	− ·0906	− ·3905	·1175
r_4	− ·2254	− ·4968	− ·1745	− ·1259	·4133	− ·1206	− ·1358
r_5	− ·1131	− ·4001	− ·0172	·1342	− ·0759	·0665	− ·3379
	15	16	17	18	19	20	21
r_1	−0·2252	0·7339	0·6321	0·4980	0·7223	0·1592	0·0173
r_2	− ·3146	·5592	·4469	·4072	·4125	·2366	·1851
r_3	·2079	·4748	·5124	·3315	·1694	− ·2656	·0154
r_4	− ·0308	·4532	·3605	·3370	− ·0076	− ·1629	− ·1446
r_5	− ·0193	·3096	·0007	− ·0459	·1146	− ·1841	− ·4016
	22	23	24	25	Result over the whole data		
r_1	0·1452	0·2553	0·3225	−0·0381	0·3094		
r_2	·0575	·3401	·1022	− ·3880	·1320		
r_3	− ·1054	·5352	·3282	·1047	·0689		
r_4	− ·1034	·1018	− ·0272	·3367	·0559		
r_5	− ·0174	·2028	− ·2516	− ·0240	− ·0360		

Table 5·2. *Serial correlations along the rows*

	1	2	3	4	5	6
r_1	−0·0678	0·0530	0·1019	−0·1860	−0·0458	0·1407
r_2	− ·1137	− ·2164	− ·1857	− ·2199	− ·4065	·0273
r_3	− ·1337	− ·2762	− ·1595	·2116	− ·1180	·0175
r_4	− ·0149	− ·1962	− ·0053	− ·2070	·1966	·1734
r_5	·1125	·0179	·0703	− ·0528	− ·3602	·1368

	7	8	9	10	11	12
r_1	0·2213	0·0092	−0·2370	0·1229	0·5122	−0·1298
r_2	− ·0176	− ·4525	− ·3271	− ·0223	·1265	− ·2140
r_3	·3199	·0435	·3571	− ·1087	− ·0216	− ·0866
r_4	− ·0091	− ·0892	·0594	− ·1608	− ·4286	− ·0825
r_5	− ·3077	− ·1494	− ·3239	− ·2848	− ·6254	·1958

	13	14	15	16	17	18
r_1	0·2216	0·2350	0·1897	0·3144	0·3287	0·5012
r_2	·0245	− ·0754	·1852	·3682	·1497	·4579
r_3	− ·0321	·0997	·3053	·2172	− ·0725	·1215
r_4	− ·1850	·1292	·0130	·0672	− ·1727	− ·0312
r_5	− ·1199	·0896	·2041	− ·1216	− ·4683	− ·1399

	19	20	Result over the whole data	
			Modified	Original
r_1	0·6305	0·1853	0·1557	0·2544
r_2	·3417	·2093	− ·0166	·0941
r_3	− ·0052	− ·1206	·0372	·1282
r_4	− ·1247	·0575	− ·0554	·0222
r_5	− ·3147	− ·3051	− ·1559	− ·0560

the same over the whole of the modified and the original data, and these are shown in the last column of the table. The correlations exhibited in the above two tables are very small, and hence the degenerate Markoff process of the first order, with the correlation function of the type $\rho_{st} = \rho_1^{|s|}\rho_2^{|t|}$, seems to be quite a satisfactory fit to the modified data. We take $\rho_1 = 0{\cdot}1557$ and $\rho_2 = 0{\cdot}3094$ as the estimates of the population values.

We fit a normal curve to the modified data. The various plots are grouped into eleven classes according to the weights of the grain, at intervals of 0·2 lb. The expected frequencies in the various groups are calculated by estimating the mean and the variance from the data. The χ^2 is given to be 4·0787. The mean and variance of χ^2 can be obtained from (11). There are $k\,(=11)$ classes, and on the assumption that the parameters are known *a priori*, we have obtained the constant factor $k-1$ in (11). However, in this particular example the two parameters m and σ^2 were estimated to find the expected frequencies in the various classes.

Following the independent case one could approximately subtract 2 from the expected value of χ^2 and 4 from the variance of χ^2. It is shown in § 6 that by estimating the parameters by the minimum χ^2 method, not only the constant factor is affected but also the terms in ρ_1. However, as this estimation problem is not discussed until § 6, in this section we provisionally make a crude reduction in the constant factor only. Hence, taking $k-3$ for $k-1$ in (11) we get

$$E(\chi^2) \sim 9{\cdot}60, \quad V(\chi^2) \sim 28{\cdot}04,$$

and the modified χ^2 is 2·79 with 6·57 degrees of freedom, which is insignificant, and hence on the basis of the above test we conclude that the normal curve fits the data.

6. Estimation of parameters

The effect of the estimation of the parameters on the modified χ^2 test is still under investigation, and it is hoped to publish the results in detail later on. Here we merely mention (without proof) some of the results so far obtained.

The minimum χ^2 estimates may not necessarily be the same as the maximum-likelihood estimates when the sequence of observations is dependent and ordered.

However, the former are easy to find in practice and are obtained by maximizing

$$M = \sum_i n_i \log m_i.$$

We shall consider the effect of these estimates only on the average value of χ^2. If we write $\check{\chi}^2$ to be the value of χ^2 when the minimum χ^2 estimates are substituted to find the expected frequencies, then for the case of one parameter θ involved in m_i

$$E(\chi^2) - E(\check{\chi}^2) \sim \frac{E(\partial M/\partial\theta)^2}{-E(\partial^2 M/\partial\theta^2)}. \tag{12}$$

For the case of two parameters θ_1 and θ_2 we obtain

$$E(\chi^2) - E(\check{\chi}^2) \sim -\frac{V_{11}M_{22} + V_{22}M_{11} - 2V_{12}M_{12}}{M_{11}M_{22} - M_{12}^2}, \tag{13}$$

where

$$V_{ij} = E\left(\frac{\partial M}{\partial\theta_i}\frac{\partial M}{\partial\theta_j}\right), \quad M_{ij} = E\left(\frac{\partial^2 M}{\partial\theta_i\partial\theta_j}\right).$$

For the Poisson-Markoff process the minimum χ^2 estimate of the mean ν is simply the sample mean (it turns out to be the maximum likelihood estimate also). The second member of (12) is approximately $(1+\rho_1)/(1-\rho_1)$, and hence $(6a)$ becomes

$$E(\check{\chi}^2) \sim k - 1 + 2\rho_1^2 + 2\rho_1^3 + O(\rho_1^4).$$

In the case of the normal Markoff process, the minimum χ^2 estimates of the mean and variance are the sample mean and variance. The second member of (13) is simply

$$\frac{1+\rho_1}{1-\rho_1} + \frac{1+\rho_1^2}{1-\rho_1^2}$$

for the one-dimensional process; and

$$\frac{1+\rho_1}{1-\rho_1}\frac{1+\rho_2}{1-\rho_2} + \frac{1+\rho_1^2}{1-\rho_1^2}\frac{1+\rho_2^2}{1-\rho_2^2}$$

for the two-dimensional process with $\rho_{st} = \rho_1^{|s|}\rho_2^{|t|}$. Hence (8*a*) and (11*a*) become respectively

$$E(\tilde{\chi}^2) \sim k - 3 + 2\sum_{s\geqslant 3} \frac{\rho_1^s}{1-\rho_1^s},$$

$$E(\tilde{\chi}^2) \sim k - 3 + 2\sum_{r\geqslant 3} \frac{\rho_1^r + \rho_2^r}{(1-\rho_1^r)(1-\rho_2^r)}.$$

Thus not only the constant term is reduced but also the terms due to the correlation coefficient, and hence we can apply the classical χ^2 test with more confidence in this case (at least for small ρ_1).

For the Mercer & Hall data $E(\tilde{\chi}^2) \sim 8{\cdot}07$ observed $\chi^2 = 4{\cdot}0787$, and with the classical χ^2 test the degrees of freedom for this are 8; and hence χ^2 is insignificant.

7. Summary

The effect of the serial dependence of observations on the classical χ^2 test is considered. For large correlations it is necessary to make the corrections to the mean and variance of χ^2 given in this paper. The theory is illustrated with particular reference to the Poisson and normal Markoff processes.

It is a pleasure to record my sincere thanks to Prof. M. S. Bartlett, under whose supervision this work was done. To him I owe the suggestion of the problem, and his comments and advice were invaluable during the investigation. I am indebted to Mrs Linnert for the calculations in Tables 5·1 and 5·2.

REFERENCES

Bartlett, M. S. (1951). *Proc. Camb. Phil. Soc.* **47**, 86.
Chandrasekhar, S. (1943). *Rev. Mod. Phys.* **15**, 1.
Good, I. J. (1953). *Proc. Camb. Phil. Soc.* **49**, 276.
Kendall, M. G. (1949). *Biometrika*, **36**, 267.
Mercer, W. B. & Hall, A. D. (1911). *J. Agric. Sci.* **4**, 107.
Neall, P. S. (1952). A χ^2 test for frequency data of simple Markoff chain processes (unpublished article).
Pearson, K. (1900). *Phil. Mag.* 5th ser. **50**, 157.
Quenouille, M. H. (1949). *Ann. Math. Statist.* **20**, 355.

APPENDIX

The following table gives the values of the expressions $\sum_s \rho_1^s/(1-\rho_1^s)$ and $\sum_s \rho_1^s/(1-\rho_1^s)^2$ involved in (8*a*) and (8*b*) for various values of ρ_1.

Table giving values of the expressions involved in (8*a*) *and* (8*b*) (*correct up to three places*)

ρ_1	$\sum_s \rho_1^s/(1-\rho_1^s)$	$\sum_s \rho_1^s/(1-\rho_1^s)^2$
0·1	0·122	0·135
0·2	0·302	0·366
0·3	0·567	0·761
0·4	0·969	1·455
0·5	1·607	2·744
0·6	2·691	5·367
0·7	4·756	11·570
0·8	9·557	30·836
0·9	27·086	143·477

THE DISTRIBUTION OF RANGE IN CERTAIN NON-NORMAL POPULATIONS

By H. A. DAVID

Commonwealth Scientific and Industrial Research Organization, Sydney, Australia

1. Introduction

The range has long been used as a measure of dispersion in samples from a normal population, and tables giving the moments and percentage points of its sampling distribution have appeared in many publications, particularly in those concerned with quality control. Comparatively little is known, however, about the distribution of range when the parent population is non-normal, although some useful information based largely on E. S. Pearson's empirical sampling investigations was reproduced in Tables XXIII and XXIV of *Tables*

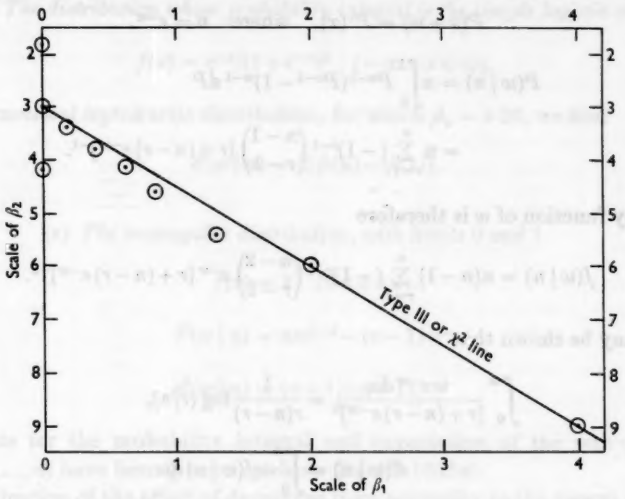


Fig. 1.

for Statisticians and Biometricians, Part 2 (K. Pearson, 1931). In the present paper we shall derive some exact results regarding the expectation and probability integral of range in samples from a number of populations, mostly asymmetrical, whose β_1, β_2 points are shown in Fig. 1. Some of these results are obtained from explicit expressions for the probability integral $P(w \mid n)$ and expectation $\mathscr{E}(w \mid n)$ of a range, w, in a sample of n. Others are derived from existing tables calculated for another purpose and based on quadrature.

2. Expressions for $P(w \mid n)$ and $\mathscr{E}(w \mid n)$ in certain special cases

It is easy to derive a formula for $P(w \mid n)$ which is valid for any continuous parental population with frequency function $f(x)$. If, without loss of generality, $f(x)$ is defined for $-\infty \leqslant x \leqslant \infty$, the formula is (see, for example, Hartley, 1942)

$$P(w \mid n) = n\int_{-\infty}^{\infty} f(x)\left(\int_{x}^{x+w} f(y)\,dy\right)^{n-1} dx$$

$$= n\int_{-\infty}^{\infty} f(x)\,(P(x+w) - P(x))^{n-1}\,dx, \qquad (1)$$

where $P(x)$ is the probability integral of $f(x)$. $\mathscr{E}(w \mid n)$ may be obtained directly from $P(w \mid n)$ or by the use of a formula given by Tippett (1925).

We shall now consider a series of examples.

(a) The distribution of $x = -\log \frac{1}{2}\chi^2$ for degrees of freedom $\nu = 2$

This is the distribution

$$f(x) = \exp(-x - e^{-x}) \quad (-\infty \leqslant x \leqslant \infty), \tag{2}$$

which was derived by Fisher & Tippett (1928) in their study of the limiting forms of the distributions of the largest or smallest members of a sample. We have

$$P(x) = \exp(-e^{-x})$$

and

$$P(x+w) = P^v(x), \quad \text{where} \quad v = e^{-w}.$$

Hence

$$P(w \mid n) = n\int_0^1 P^{n-1}(P^{v-1}-1)^{n-1}\,dP$$

$$= n\sum_{r=1}^{n}(-1)^{r-1}\binom{n-1}{r-1}[r+(n-r)e^{-w}]^{-1}.$$

The frequency function of w is therefore

$$f(w \mid n) = n(n-1)\sum_{r=1}^{n}(-1)^{r-1}\binom{n-2}{r-1}e^{-w}[r+(n-r)e^{-w}]^{-2}.$$

Further, it may be shown that

$$\int_0^\infty \frac{we^{-w}\,dw}{[r+(n-r)e^{-w}]^2} = \frac{1}{r(n-r)}\log(r/n),$$

and hence

$$\mathscr{E}(w \mid n) = \int_0^\infty wf(w \mid n)\,dw$$

$$= \sum_{r=1}^{n}(-1)^r\binom{n}{r}\log(r/n).$$

(b) The exponential distribution, or distribution of $\frac{1}{2}\chi^2$ with degrees of freedom $\nu = 2$

$$f(x) = e^{-x} \quad (0 \leqslant x \leqslant \infty). \tag{3}$$

Here

$$P(w \mid n) = (1-e^{-w})^{n-1},$$

$$\mathscr{E}(w \mid n) = \psi(n) - \psi(1) = \sum_{s=1}^{n-1} s^{-1},$$

where

$$\psi(n) = d\log\Gamma(n)/dn.$$

It is also easy to obtain $P(w \mid n)$ for the case

$$f(x) = ae^{-x} + be^{-cx} \quad (0 \leqslant x \leqslant \infty). \tag{4}$$

(c) *The distribution of* $\frac{1}{2}\chi^2$ *with degrees of freedom* $\nu = 4$

$$f(x) = x e^{-x} \quad (0 \leqslant x \leqslant \infty), \tag{5}$$

$$P(w \mid n) = u^{n-1} \sum_{r=1}^{n} \binom{n-1}{r-1} \frac{r!}{n^r} v^{r-1},$$

where $$u = 1 - e^{-w} - w e^{-w}, \quad uv = 1 - e^{-w},$$

$$\mathscr{E}(w \mid n) = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \sum_{r=0}^{s} \frac{s(s-1)\dots(s-r+1)}{s^{r+1}} - \sum_{r=0}^{n} \frac{n(n-1)\dots(n-r+1)}{n^{r+1}}.$$

(d) *The distribution whose probability integral is the simple logistic curve*

$$f(x) = e^{-x}/(1+e^{-x})^2 \quad (-\infty \leqslant x \leqslant \infty). \tag{6}$$

For this symmetrical leptokurtic distribution, for which $\beta_2 = 4{\cdot}20$, we find

$$\mathscr{E}(w \mid n) = 2[\psi(n) - \psi(1)].$$

(e) *The rectangular distribution, with limits* 0 *and* 1

$$f(x) = 1 \quad (0 \leqslant x \leqslant 1). \tag{7}$$

Here $$P(w \mid n) = n w^{n-1} - (n-1) w^n,$$

$$\mathscr{E}(w \mid n) = (n-1)/(n+1).$$

Similar results for the probability integral and expectation of the mth-order statistic x_m $(m = 1, 2, \dots, n)$ have been given elsewhere (David, 1952*a*).

A good indication of the effect of departure from normality in the parent distribution is obtained by determining the probability that the range exceeds the 'normal theory' significance points. Thus if $k(n, \alpha)$ is the $100\alpha\,\%$ significance point for the standardized range $W = w/\sigma_x$ in samples of n from a normal population with standard deviation σ_x, it is necessary to calculate for the distributions given above, $Q(w \mid n) = 1 - P(w \mid n)$, where $w = k(n, \alpha)\,\sigma_x$. The last three columns of Table 1 give such results for the parent populations of equations (2), (5) and (3) respectively, taking $\alpha = 0{\cdot}05$, $0{\cdot}01$ and $n = 2\,(2)\,12$. The values of $k(n, \alpha)$ were obtained from Pearson & Hartley's (1954) Table 22. In the case of the rectangular distribution of equation (7), it is impossible for the range to fall beyond the normal theory significance points for the cases chosen, except for $n = 2$, $\alpha = 0{\cdot}05$ when the probability is $0{\cdot}04$.

If d_n denotes the expected value of the standardized range $W = w/\sigma_x$ in samples of n from a normal population, it is common to use w/d_n as an estimate of σ_x because $\mathscr{E}(w/d_n) = \sigma_x$ for normal samples. Table 2 gives value of $\mathscr{E}(w/(d_n \sigma_x))$ for $n = 2\,(2)\,12$ and for the populations of equations (7), (6), (2), (5) and (3) respectively, thus indicating the amount of bias involved in the current procedure of estimating a standard deviation from a range or mean range

using the corrective factor appropriate for normal distributions. Plackett (1947) has shown that the expectation of the ratio $w/(d_n\sigma_x)$ has a finite maximum given by

$$n\{2[(2n-2)!-((n-1)!)^2]/(2n-1)!\}^{\frac{1}{2}}.$$

This is given in the last column of the table, and is attained when the inverse to $P(x)$ is of the form $x(P) = \text{const.}\{P^{n-1}-(1-P)^{n-1}\}$.

Table 1. *Showing the probability that the standardized range in samples of n from various non-normal distributions falls beyond the normal theory* 5 *and* 1 % *significance levels* ($\alpha = 0{\cdot}05$ *and* $0{\cdot}01$)

		Distributions of $\log \chi^2$					Distributions of χ^2	
ν (for χ^2) ...		12	6	4	3	2	4	2
β_1 ...		0·18	0·39	0·61	0·84	1·30	2·00	4·00
β_2 ...		3·36	3·76	4·19	4·61	5·40	6·00	9·00
Equation no. ...		—	—	—	—	(2)	(5)	(3)
α	n							
0·05	2	0·051	0·052	0·053	0·054	0·056	0·059	0·063
	4	·054	·058	·060	·063	·067	·070	·077
	6	·056	·061	·065	·071	·075	·076	·086
	8	·058	·064	·070	·076	·081	·080	·092
	10	·059	·067	·074	·081	·087	·085	·098
	12	·061	·069	·077	·086	·092	·088	·104
0·01	2	0·012	0·014	0·015	0·016	0·019	0·021	0·026
	4	·013	·017	·019	·021	·026	·028	·036
	6	·014	·019	·022	·023	·030	·031	·042
	8	·014	·020	·025	·025	·034	·034	·047
	10	·015	·021	·027	·027	·037	·037	·050
	12	·016	·022	·028	·028	·040	·039	·054

Table 2. *Expectation of* $w/(d_n\sigma_x)$, *the ratio of the standardized range to its normal theory expected value*

Type of population $f(x)$	Rectangular	Logistic	$f(x)=\exp(-x-e^{-x})$	$f(x)=xe^{-x}$	Exponential	Maximum value
Equation no. ...	(7)	(6)	(2)	(5)	(3)	—
n						
2	1·023	0·978	0·958	0·940	0·886	1·023
4	1·019	0·982	0·961	0·942	0·891	1·012
6	0·976	0·993	0·968	0·947	0·901	1·007
8	0·946	1·004	0·975	0·951	0·911	1·026
10	0·921	1·014	0·980	0·955	0·919	1·054
12	0·900	1·022	0·986	0·959	0·927	1·086

3. EVALUATION OF $P(w \mid n)$ BY NUMERICAL INTEGRATION FOR DISTRIBUTIONS OF $x = \log \chi^2$

In cases where no simple explicit expression for $P(w \mid n)$ seems possible, direct numerical integration at $w = k(n, \alpha)\sigma_x$ could be attempted by means of equation (1). The procedure is in general likely to be tedious, but for one type of distribution calculations for tables recently computed provide a quick answer. The present writer (David, 1952*b*) computed a table (for normal variation) of the upper 5 and 1 % points of the ratio $s^2_{\max.}/s^2_{\min.}$ of the largest to the smallest mean square in a set of k independent mean squares s_t^2 $(t = 1, \ldots, k)$, each based on ν degrees of freedom. Clearly the logarithm of this ratio is the range in a sample of k independent values of $x = \log \chi^2$, where ν is the degrees of freedom of χ^2.

From the quadratures yielding the framework of the $s^2_{\max.}/s^2_{\min.}$ table of percentage points, it was possible to find $Q(w \mid n) = 1 - P(w \mid n)$ for $w = k(n, \alpha)\sigma_x$ as in the previous section, and these values are included in Table 1 for $\nu = 12, 6, 4, 3, 2$. For $\nu = 2$ an explicit solution is possible and has been discussed already. The beta-coefficients for the distribution of $x = \log \chi^2$, given at the head of the columns, are taken from Bartlett & Kendall (1947) who have studied these distributions in some detail.

It will be seen that $s^2_{\max.}/s^2_{\min.}$ is a special case of what Gumbel & Herbach (1951) have termed the 'extremal quotient', i.e. the ratio of the largest to the smallest of a set of independent variates. The authors confine themselves in their applications to the case $n = 2$. Clearly all the results of § 2 regarding the probability integral of range have immediate counterparts for the extremal quotient.

4. GENERAL COMMENT ON THE RESULTS

The results given in Tables 1 and 2 give a broad picture of the way in which the distribution of range is affected by departure from normality, when the parent distribution is roughly of χ^2 or Pearson Type III form. The exact values for $\mathscr{E}(w/(d_n\sigma_x))$ in Table 2 may be extended by reference to the empirical results based on random sampling experiments summarized by E. S. Pearson (1950). Remembering that the rectangular and exponential distributions, and even the distribution $f(x) = xe^{-x}$ with $\beta_1 = 2$, $\beta_2 = 6$ are exceptional forms, it will be seen that the expectation is remarkably insensitive to changes in the population β_1 and β_2 values. Thus the estimates of standard deviation obtained in industrial quality control by dividing a mean range based on m small samples of size n is unlikely to be seriously biased and therefore will provide satisfactory control limits for a mean.

Table 1 shows, however, that the position will be rather less satisfactory when we are concerned with providing control limits for the range itself. This arises because the standard deviation of range, w, just as that of the root mean square estimator, s, is a function of the population β_2, very roughly varying as $\sqrt{(\beta_2 - 1)}$. For the asymmetrical parent distributions dealt with in Table 1, use of the normal theory control limits will therefore tend to make the statistician look for trouble in the shape of excessive variability when in fact the fault is departure from normality. This may not be altogether a bad thing. The effect increases with increasing sample size and is relatively greater, the more extreme the significance level used.

5. COMPARISON OF THE EFFECT OF NON-NORMALITY ON THE TAIL PROBABILITIES OF w AND s

So far we have confined ourselves to a consideration of the effect of non-normality on the distribution of w. It is of obvious interest to compare this effect with that on the distribution of s. Such a comparison has been made by experimental sampling for three empirical

populations (Pearson & Haines, 1935). The authors found, for $n = 5$ and 10, that the number of samples out of 500 falling beyond upper and lower normal theory 5 and 0·5 % levels was not significantly different for w than for s.

A theoretical comparison is difficult, primarily because of the complexity of the distribution of s for non-normal universes. The approach tried was to use Gayen's (1950) Table 4, which gives the corrective functions for determining the tail probabilities of the variance ratio at the upper 5 % normal theory significance levels. The present case corresponds to the test for compatibility of two variances in independent samples, one of which is very large, drawn from a population with common β_1 and β_2. Thus, in Gayen's notation we use v with $\nu_2 = \infty$, $\lambda_3'' = \lambda_3'$, $\lambda_4'' = \lambda_4'$. Comparisons of tail probabilities of w and s at their upper 5 % normal theory values are set out below for three of the distributions of $x = \log \chi^2$ given in Table 1.

Table 3. *Probability that s and w fall beyond normal theory 5 % levels when the variable is distributed as* $\log \chi^2$ *with* ν *degrees of freedom*

Sample size, n	$\nu = 12$		$\nu = 6$		$\nu = 4$	
	s	w	s	w	s	w
2	0·051	0·051	0·053	0·052	0·055	0·053
4	0·058	0·054	0·066	0·058	0·075	0·060
6	0·060	0·056	0·072	0·061	0·084	0·065

The results are interesting but the imperfect agreement for $n = 2$ (when w and s are equivalent) is an indication that the Edgeworth's series approximation employed by Gayen is here not very satisfactory. The discrepancy is greater still for the other distributions of Table 1.

I am grateful to Prof. E. S. Pearson, for suggesting a number of valuable improvements to the first draft of this paper.

REFERENCES

Bartlett, M. S. & Kendall, D. G. (1946). *J. R. Statist. Soc. Suppl.* **8**, 128.
David, H. A. (1952*a*). Unpublished Ph.D. Thesis. University of London.
David, H. A. (1952*b*). *Biometrika*, **39**, 422.
Fisher, R. A. & Tippett, L. H. C. (1928). *Proc. Camb. Phil. Soc.* **24**, 180.
Gayen, A. K. (1950). *Biometrika*, **37**, 236.
Gumbel, E. J. & Herbach, L. H. (1951). *Ann. Math. Statist.* **22**, 418.
Hartley, H. O. (1942). *Biometrika*, **32**, 334.
Pearson, E. S. (1950). *Biometrika*, **37**, 88.
Pearson, E. S. & Haines, Joan (1935). *J. R. Statist. Soc. Suppl.* **2**, 83.
Pearson, E. S. & Hartley, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge: The University Press for the Biometrika Trust.
Pearson, K. (1931). *Tables for Statisticians and Biometricians*, Part 2. Cambridge: The University Press for the Biometrika Trust.
Plackett, R. L. (1947). *Biometrika*, **34**, 120.
Tippett, L. H. C. (1925). *Biometrika*, **17**, 364.

THE MEAN AND COEFFICIENT OF VARIATION OF RANGE IN SMALL SAMPLES FROM NON-NORMAL POPULATIONS

By D. R. COX

Statistical Laboratory, University of Cambridge

By examining special populations, a table is obtained for predicting approximately the mean and coefficient of variation of the range of random samples of n ($n \leqslant 5$), drawn from a population of specified kurtosis, β_2. The effect of non-normality on various statistical methods that use the range is then considered.

1. Introduction

Since Tippett (1925) tabulated the mean range of random samples from a normal population, the range has been extensively used for the rapid estimation of dispersion. Moreover, much work has been done recently on quick significance tests in which the root-mean-square estimate of dispersion in the t-test, analysis of variance, etc., is replaced by an estimate derived from the range. All these uses of the range rest on the assumption of normality, and so it is of interest to examine the distribution of range from non-normal populations. This was first done by E. S. Pearson & Adyanthaya (1929), and their work, and later work by Shone (1949), has been summarized, discussed and extended by Pearson (1950). The conclusion was that in small samples the ratio of mean range to population standard deviation is not much affected by the form of the population, but that the coefficient of variation of range depends fairly critically on the population. In large samples the distribution of range is, of course, determined by the tails of the population and so is very sensitive to non-normality.

The main object of the present paper is to predict the mean and coefficient of variation of range in small random samples of n ($n \leqslant 5$) from a population of given skewness and kurtosis and then to show how these results can be used to assess the effect of non-normality on the common applications of the range. Such a prediction can only be approximate because there is no functional relation between the distribution of range and population skewness and kurtosis. There are four ways of proceeding:

(i) by evaluating numerically the single and double integrals for the mean and mean square of range for a representative selection of non-normal populations;

(ii) by sampling experiments;

(iii) by a derivation of upper and lower limits for the mean and coefficient of variation of range of populations with given properties. This was first done by Plackett (1947), who obtained an upper bound to the ratio of mean range to population standard deviation in samples of given size from an arbitrary population. An important extension of Plackett's work (Hartley & David, 1954) appeared as the present paper was being completed;

(iv) by finding a set of populations with widely varying β_1, β_2 for which the properties of range can be expressed exactly or approximately in terms of known quantities.

While (iii) is the most satisfying approach from a mathematical point of view, method (iv) seems to lead more directly to a working answer to the problem and it will be followed here.

2. Mixed normal populations

Consider a population formed by combining in proportions $1-p$, p normal distributions of zero mean and unit standard deviation and of positive mean μ and standard deviation τ. We call such a population an *unsymmetrical mixed* population.* The drawing of an observation from such a population can be thought of as a two-stage process, first the selection with probabilities $1-p$, p of a normal distribution and then the drawing of an observation from the appropriate normal distribution. We shall assume that μ, τ are such that in samples of the size considered no observation from the second distribution falls below an observation from the first distribution, i.e. that the constituent normal distributions are effectively non-overlapping. It is then easy to write down the mean and mean square of the range of random samples of n from the mixed population. For let d_r, e_r be the mean and mean square of range in samples of r from the unit normal population (Pearson & Hartley, 1951) and $m_r = \frac{1}{2}d_r$, n_r the mean and mean square of the largest value of samples of r from the unit normal population (Karl Pearson, 1931, Table XXI).

To illustrate the method, let $n = 3$; then Table 1 shows the various possibilities. Thus in the second line the probability that only one observation is taken from the upper distribution is given by the binomial expression $3(1-p)^2 p$, and given the allocation of observations, the range is the difference of a single observation from the upper distribution and the smaller of two observations from the unit normal distribution. The expressions for the conditional mean and mean-square range follow immediately. The overall mean range is the sum of products of the third and fourth columns of Table 1, and the overall mean-square range is the sum of products of the third and fifth columns.

Table 1. *Samples of three from unsymmetrical mixed populations*

No. of observations in lower distribution	No. of observations in upper distribution	Probability	Conditional mean range	Conditional mean square range
3	0	$(1-p)^3$	d_3	e_3
2	1	$3(1-p)^2 p$	$\mu + m_2$	$\mu^2 + 2\mu m_2 + \tau^2 + n_2$
1	2	$3(1-p)\, p^2$	$\mu + m_2\tau$	$\mu^2 + 2\mu m_2\tau + \tau^2 m_2^2 + 1$
0	3	p^3	$d_3\tau$	$e_3\tau^2$

Similar formulae can be obtained for other values of n; there is no point in writing down a general formula. More refined formulae making allowance for the overlap of the two distributions can be found. They will not be given here, although they have been used to check that overlap produces no appreciable error in the conclusions set out below.

We can deal in the same way with *symmetrical mixed* populations, i.e. mixtures in proportion $1-2p$, p, p of the unit normal distribution with normal distributions of means μ, $-\mu$ and standard deviation τ. Again we take μ, τ so that the constituent distributions are effectively non-overlapping. The mean and mean square of range can be obtained by a method similar to that used in Table 1, the details being a little more complicated.

For both types of mixed population, the mean, standard deviation, β_1 and β_2 can be worked out in a straightforward way. The mean and coefficient of variation of range have

* The use of mixed normal populations is by no means new.

been found for samples of size $n = 2, 3, 4, 5$ from the following mixed populations: $\mu = 3$, $\tau = 0$; $\mu = 4$, $\tau = 0, \frac{1}{2}$; $\mu = 5$, $\tau = 0, \frac{1}{2}, 1$, taken in combination with $p = 0{\cdot}01, 0{\cdot}05, 0{\cdot}1, 0{\cdot}2, 0{\cdot}5$. Each combination of p, μ, τ gives two mixed populations, one symmetrical and one unsymmetrical. The value of β_2 goes from 1 to 8·2 and the value of β_1 from 0 to 2·2.

The main justification for the use of mixed normal populations is that they form a sufficiently flexible set of manageable populations. However, some types of non-normality occurring in practice can be represented by a single normal population contaminated by occasional 'outliers'. If the outliers have a normal distribution, a mixed normal population results.

3. Some mathematical results

The numerical work outlined in § 2 will be the main basis for the proposed relation between the properties of range and the form of the population, but it is very desirable to supplement this work by results for populations differing as much as possible from the mixed normal form.

In random samples of n from a population with frequency function $p(x)$ and distribution function $P(x)$, the probability density of range, $f_n(w)$, is given by

$$f_n(w) = n(n-1)\int_{-\infty}^{\infty} p(x)\,p(x+w)\,[P(x+w)-P(x)]^{n-2}\,dx. \tag{1}$$

Consider the exponential population with $p(x) = e^{-x}$ $(x \geqslant 0,\ p(x) = 0,\ x < 0)$. Then (1) gives (Maguire, Pearson & Wynn, 1952)

$$f_n(w) = (n-1)\,(1-e^{-w})^{n-2}\,e^{-w}, \tag{2}$$

and the moment-generating function of range, $M_n(t)$, is

$$\begin{aligned} M_n(t) &= \int_0^{\infty} e^{tw} f_n(w)\,dw \\ &= \Gamma(n)\,\Gamma(t+1)/\Gamma(n+t). \end{aligned} \tag{3}$$

The cumulants of range are obtained by taking logs and expanding in powers of t, so that the mean and variance of range are $\sum_{r=1}^{n-1} 1/r$ and $\sum_{r=1}^{n-1} 1/r^2$, on using well-known properties (Jeffreys & Jeffreys, 1946) of the digamma and trigamma functions. The above exponential population has standard deviation $\sigma = 1$ and $\beta_1 = 4$, $\beta_2 = 9$.

Similar calculations can be done for populations with frequency functions $x^r e^{-x}/r!$, where r is a small integer, although the results get rapidly more cumbersome as r, n increase. A few results for the population $r = 1$, $\sigma = \sqrt{2}$, $\beta_1 = 2$, $\beta_2 = 6$, are given in Table 2, together with corresponding results for the exponential population, and, for comparison, the standard values for a normal population.*

Plackett (1947) showed that the ratio of mean range to the population standard deviation σ can, for given n, be made arbitrarily small by choice of the population. It is of interest to examine in more detail the conditions under which this ratio is zero. Now if the distribution function of the population is $P(x)$, the distribution functions of the largest and smallest values in a random sample of n are respectively $P^n(x)$ and $1-(1-P(x))^n$. Since the range is the difference between the largest and smallest values, it follows that the mean

* [These two population distributions are also considered, among others, by H. A. David earlier in this issue of *Biometrika* (see pp. 463–8)—Ed.].

range, $E_n(w)$, is the difference between the corresponding mean values and so is given by the Stieltjes integral

$$E_n(w) = \int_{-\infty}^{\infty} x d\{P^n(x) + (1-P(x))^n - 1\}. \tag{4}$$

This converges at $-\infty$ if and only if $\int_{-\infty} x dP(x)$ converges, and at $+\infty$ if and only if $\int^{\infty} x d(1-P(x))$ converges, i.e. (4) converges if and only if $\int_{-\infty}^{\infty} x dP(x)$ converges.* Therefore the mean range is finite if and only if the population mean is finite.

Table 2. *Properties of range of samples of n from populations* (*a*) e^{-x}, (*b*) xe^{-x}, (*c*) *normal*

n	Population (*a*)		Population (*b*)		Population (*c*)	
	Mean range/σ	c.v. of range	Mean range/σ	c.v. of range	Mean range/σ	c.v. of range
2	1	1	1·061	0·882	1·128	0·756
3	1·5	0·746	1·591	0·640	1·693	0·525
4	1·833	0·636	1·961	—	2·059	0·427
5	2·083	0·573	2·252	—	2·326	0·372

In this case $\lim_{x\to\infty} x(1-P(x)) = \lim_{x\to-\infty} xP(x) = 0$,

and we may integrate (4) by parts to give

$$E_n(w) = \int_{-\infty}^{\infty} \{1 - P^n(x) - (1-P(x))^n\} dx. \tag{5}$$

This is the form used by Tippett in tabulating the mean range; the present proof is considerably shorter than the one current in the literature (Kendall, 1943).

We have proved, therefore, that there are many populations, in particular those with finite mean and infinite variance, for which $E_n(w)/\sigma$ is zero. Further, such a population should be considered as having infinite β_2. For although β_2 is strictly not defined, the original population is for practical purposes equivalent to one truncated at $\pm X$, where X is very large, and such a truncated population has a very large β_2. Hence for some sets of populations $E_n(w)/\sigma$ tends to zero as β_2 tends to infinity, although this is not true in full generality.

4. Numerical results

We shall discuss in detail the results for $n = 5$, since this enables earlier work summarized by Pearson (1950) to be used. Fig. 1 gives mean range/$(d_5\sigma)$, where d_5 is Tippett's divisor for samples of five, plotted against β_2. As indicated in the legend, results for five types of population are included in the diagram:

(i) theoretical; rectangular ($\beta_2 = 1{\cdot}8$), normal ($\beta_2 = 3$), xe^{-x} ($\beta_2 = 6$) and exponential ($\beta_2 = 9$);

(ii) symmetrical mixed normal populations;

(iii) unsymmetrical mixed normal populations;

* Full convergence is required and not just the convergence of a principal value.

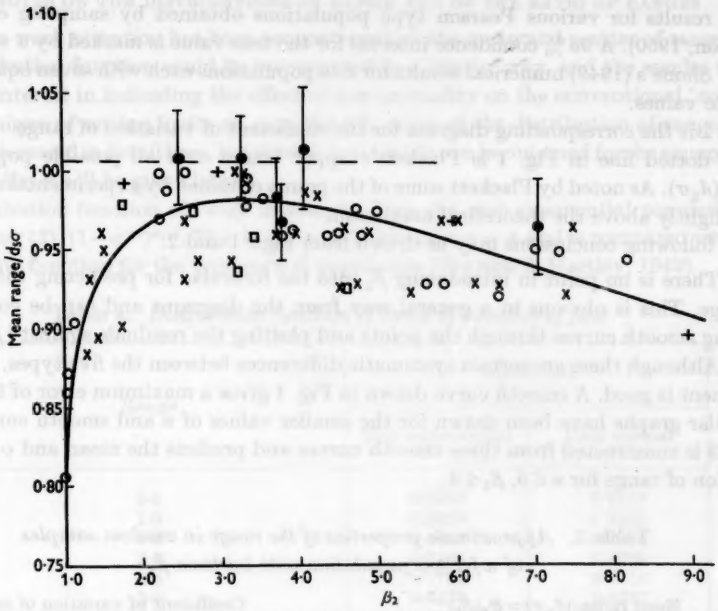


Fig. 1. Samples of five. Mean range/$(d_5\sigma)$. + theoretical. ○ symmetrical mixed normal populations. × unsymmetrical mixed normal populations. ● sampling experiments. □ Shone's discrete populations. ----- theoretical upper bound.

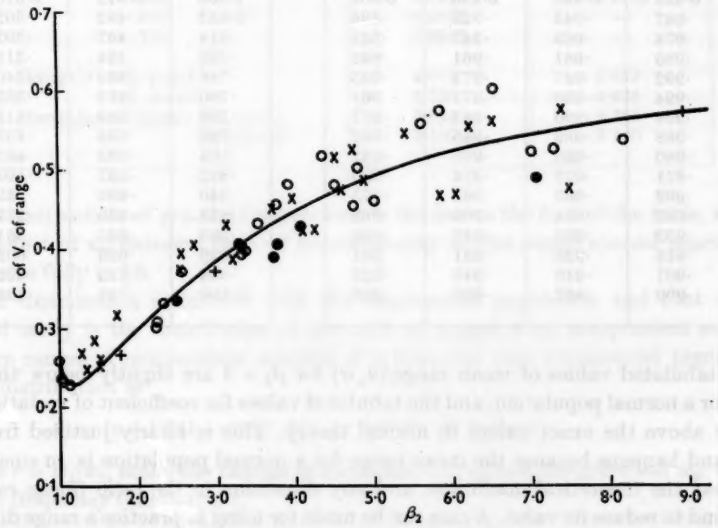


Fig. 2. Samples of five. Coefficient of variation of range. For key, see legend to Fig. 1.

(iv) results for various Pearson type populations obtained by sampling experiments (Pearson, 1950). A 95 % confidence interval for the true value is marked by a vertical line;

(v) Shone's (1949) numerical results for five populations each with seven equally spaced discrete values.

Fig. 2 is the corresponding diagram for the coefficient of variation of range.

The dotted line in Fig. 1 is Plackett's upper bound over all possible populations of $E_5(w)/(d_5\sigma)$. As noted by Plackett some of the points obtained by experimental sampling lie very slightly above the theoretical maximum.

The following conclusions may be drawn from Figs. 1 and 2:

(i) There is no point in introducing β_1 into the formulae for predicting the properties of range. This is obvious in a general way from the diagrams and can be confirmed by drawing smooth curves through the points and plotting the residuals against $|\beta_1|$.

(ii) Although there are certain systematic differences between the five types, the general agreement is good. A smooth curve drawn in Fig. 1 gives a maximum error of 5 %.

Similar graphs have been drawn for the smaller values of n and smooth curves drawn. Table 3 is constructed from these smooth curves and predicts the mean and coefficient of variation of range for $n \leqslant 5$, $\beta_2 \leqslant 9$.

Table 3. *Approximate properties of the range in random samples of n from a population with kurtosis* β_2

	Mean range$/(d_n\sigma) = B_n(\beta_2)$				Coefficient of variation of range $= C_n(\beta_2)$			
β_2	$n=2$	3	4	5	$n=2$	3	4	5
1·0	0·886	0·886	0·850	0·806	1·000	0·575	0·375	0·258
1·2	·947	·945	·923	·896	0·853	·482	·302	·233
1·4	·974	·969	·947	·924	·814	·467	·303	·242
1·6	·985	·981	·961	·942	·796	·464	·319	·260
1·8	·992	·987	·971	·955	·786	·469	·340	·280
2·0	·994	·990	·977	·964	·780	·479	·363	·300
2·5	·994	·990	·985	·977	·780	·509	·412	·344
3·0	·988	·986	·985	·982	·790	·538	·450	·384
3·5	·980	·980	·980	·982	·805	·563	·482	·419
4·0	·971	·972	·974	·980	·822	·587	·509	·447
4·5	·962	·963	·967	·975	·840	·609	·532	·470
5·0	·952	·954	·960	·969	·858	·629	·552	·490
6·0	·932	·938	·946	·956	·893	·665	·581	·521
7·0	·918	·923	·931	·941	·929	·696	·602	·542
8·0	·907	·910	·915	·925	·964	·723	·622	·560
9·0	·900	·897	·900	·909	1·000	·748	·636	·573

The tabulated values of mean range$/(d_n\sigma)$ for $\beta_2 = 3$ are slightly below the value of unity for a normal population, and the tabulated values for coefficient of variation of range slightly above the exact values in normal theory. This is clearly justified from Figs. 1 and 2 and happens because the mean range for a normal population is, in small samples, very near the theoretical maximum, and any dispersion in the ratio (mean range)$/(d_n\sigma)$ must tend to reduce its value. A case can be made for using in practice a range divisor lower than the usual d_n; this and other practical implications of Table 3 are discussed in § 6.

5. Notes on the distributions of range and of the ratio of ranges

In the above work attention has been concentrated on the mean and scatter of range. The whole distribution function could be investigated in a similar way, and the results would be of some interest in indicating the effect of non-normality on the conventional 'control-chart' technique of setting limits at, say, the 5% point of the distribution of range. This will not be discussed in detail here, but since simple results can be obtained for the exponential population, these will be given briefly.

The distribution function of range in samples from the unit exponential population is, on integrating (2), $(1-e^{-w})^{n-1}$. This is given in Table 4 for $n = 4$ and is compared with the corresponding function for the unit normal population (Pearson & Hartley, 1942).

Table 4. *Distribution function of range of samples of four*

Range	Population	
	Unit exponential	Unit normal
0·5	0·0609	0·0152
1·0	0·2525	0·1057
1·5	0·4689	0·2685
2·0	0·6465	0·5096
2·5	0·7734	0·7710
3·0	0·8579	0·8537
3·5	0·9121	0·9361
4·0	0·9461	0·9758
4·5	0·9671	0·9920
5·0	0·9800	0·9977
5·5	0·9878	0·9994
6·0	0·9925	0·9999
6·5	0·9955	—
7·0	0·9973	—
Range/σ: 5% point	4·078	3·633
1% point	5·700	4·405
Range/mean range: 5% point	2·224	1·765
1% point	3·109	2·140

The main comparison of practical importance is the one at the foot of the table, since it shows the effect of an extreme form of non-normality on the range control chart. It is discussed more fully in § 6.

A further distribution connected with the exponential population and that can be investigated easily is the distribution of the ratio of ranges from independent samples. If w_1, w_2 are ranges of independent samples of n from the unit exponential population, their joint distribution is

$$(n-1)^2(1-e^{-w_1})^{n-2}(1-e^{-w_2})^{n-2}e^{-w_1-w_2}dw_1dw_2.$$

If we define $r = w_1/w_2$, make the change of variables to r, w_2 and integrate out w_2, we find for $q(r)$, the frequency function of r,

$$q(r) = (n-1)^2\int_0^\infty (1-e^{-rw_2})^{n-2}(1-e^{-w_2})^{n-2}w_2e^{-w_2(r+1)}dw_2. \qquad (6)$$

Now it is easy to show that

$$\int_r^\infty e^{-rw_2}(1-e^{-rw_2})^{n-2}\,dr = \frac{1}{(n-1)w_2}\{1-(1-e^{-rw_2})^{n-1}\}.$$

Hence we may determine the distribution function, $Q(r)$, of r by integrating (6) under the integral sign, thus giving

$$\begin{aligned}1-Q(r) &= (n-1)\int_0^\infty (1-e^{-w_2})^{n-2}\{1-(1-e^{-rw_2})\}^{n-1}e^{-w_2}\,dw_2\\ &= (n-1)\int_0^r \{1-(1-u^r)^{n-1}\}(1-u)^{n-2}\,du,\end{aligned}$$

on putting $u = e^{-w_2}$. This is a rational function of r and for small n takes a simple form; thus for

$$n=2,\quad 1-Q(r) = r/(r+1);$$
$$n=3,\quad 1-Q(r) = (7r+2)/(r+1)(r+2)(r+3).$$

From the corresponding formula for $n = 4$, which is rather more complicated, the upper 10, 5 and 1 % points for the ratio of two ranges of four are found to be 3·3, 4·7, 9·6. The corresponding values for a normal population are (Link, 1950) 2·4, 3·1, 5·6. There is a discussion in § 6.

6. Application

We now consider the practical implications of the above results by discussing in turn four types of application of the range. The aim is to indicate the methods for investigating the effect of non-normality rather than to present comprehensive conclusions.

(i) *Point estimation of dispersion.* Suppose that there are available a large number N of samples of n observations from populations of constant shape, and that it is required to estimate the standard deviation, σ, from the variation within samples. Consider the estimates based on (*a*) the mean square within samples and on (*b*) the mean range within samples, using the usual divisor d_n.

The first estimate has bias $O(N^{-1})$, which we ignore, and approximate standard error

$$\frac{\sigma}{2\sqrt{N}}\left\{\frac{\beta_2-3}{n}+\frac{2}{n-1}\right\}^{\frac{1}{2}}. \tag{7}$$

The second has approximate bias $(B_n(\beta_2)-1)\sigma$ and approximate standard error

$$\sigma C_n(\beta_2)\,B_n(\beta_2)/\sqrt{N},$$

where $B_n(\beta_2)$, $C_n(\beta_2)$ are the quantities given in Table 3.

The first thing to look at is the relative magnitude of the bias and the standard error of the estimate based on the range; the bias is the larger if

$$N > \frac{C_n^2(\beta_2)\,B_n^2(\beta_2)}{(1-B_n(\beta_2))^2}. \tag{8}$$

The critical value of N is given for a few special cases in Table 5. Thus, except for small β_2, N has to be large before the bias is comparable with the standard error. It is fair to say that in most applications the standard error will be the larger.

Table 5. *Approximate value of N at which bias equals standard error*

n \ β_2	1·0	2·0	3·0	4·0	5·0	7·0	9·0
2	60	16000	4200	760	290	110	81
5	1	6400	440	480	230	75	33

Suppose now that N is such that this is the case and consider the ratio of the standard error of the estimate based on the range to that of the first estimate. This ratio is

$$2C_n(\beta_2)\,B_n(\beta_2)\left\{\frac{\beta_2-3}{n}+\frac{2}{n-1}\right\}^{-\frac{1}{2}}. \tag{9}$$

It is given for a few values of n and β_2 in Table 6.

Table 6. *Ratio of standard errors of estimate from range and root-mean-square estimate*

n \ β_2	1·0	2·0	3·0	4·0	5·0	7·0	9·0
2	1·77	1·27	1·10	1·01	0·94	0·85	0·80
5	1·32	1·06	1·07	1·05	1·00	0·90	0·80

The general conclusion is that for the small and moderate values of β_2 the root-mean-square estimate is the better one, but that for the larger values of β_2 the estimate based on the range, even though biased, is to be preferred. The reason is essentially that for populations with large β_2 very extreme observations are common, and these have relatively less effect on the range than on the other estimate. The conclusion only holds when N is sufficiently small for the bias to be neglected (Table 5), although if the population β_2 were known approximately, much of the bias could be removed by using a divisor $B_n(\beta_2)\,d_n$ instead of d_n.

This leads to the point noted in § 4 that the curves fitted to the mean range pass at $\beta_2 = 3$ below the theoretical point for the normal population. In fact if little were known about the population except that β_2 lay within rather wide limits, it would be sensible to use a divisor 2–5 % lower than d_n, depending on n and the range of possible values of β_2. For this would not lead to appreciable bias if the population were normal and would reduce the bias for extreme values of β_2, both large and small.

If β_2 were known one could, instead of producing an unbiased estimate, choose a divisor as a function of n, N and β_2 in order to minimize, say $E(s_w-\sigma)^2/\sigma^2$, where s_w is the estimate of σ. This would be relevant if some irrevocable decision rested on the value of the estimate and if the 'loss function' were known, but in more usual cases I do not think anything would be gained by this. The criterion of unbiasedness, while certainly arbitrary, leads to an estimate with known distributional properties and centred near the true parameteric value; the point estimate is therefore near the true value, and the confidence distribution of the unknown parameter can be found and this is all that is required.

(ii) *Control chart for ranges.* In the usual control-chart technique for ranges, the mean range is found for a preliminary run, and say 5 and 1 % limits set at multiples of the mean

range determined from the distribution function of the range in samples from the unit normal population. If the population is not normal two problems arise:

(*a*) If the population β_2 is approximately known, where should the limits be placed? To answer this, we approximate to the distribution of range either by a χ distribution (Patnaik, 1950), or by a χ^2 distribution (Cox, 1949). These approximations have, of course, only been tested for the range in *normal* samples, but are here, nevertheless, employed as approximations to range in non-normal samples. Although we have checked this approximation in the case of the exponential below this method must, as yet, be regarded as tentative. An advantage of the χ^2 approximation over that using χ is that it is rather simpler and deals more easily with mean ranges. Limits based on the χ^2 approximation will be given here; the corresponding limits for the χ approximation can be obtained similarly. Write range/(mean range) in the form χ^2_ν/ν; this assures agreement of the first moment, while to get agreement of the second moment we must take $\nu = \nu_n(\beta_2) = 2/C_n^2(\beta_2)$. Thus limits for range/(mean range) should be set at $1/\nu_n(\beta_2)$ times the appropriate percentage point of χ^2 with $\nu_n(\beta_2)$ degrees of freedom.

Example. For the exponential population $C_4(\beta_2) = 0{\cdot}636$ (Tables 2 and 3), so that $\nu = 4{\cdot}94$ and the approximate 5 and 1 % limits are at $1/4{\cdot}94 \times 10{\cdot}98$, $14{\cdot}98$, i.e. at $2{\cdot}22$ and $3{\cdot}03$, interpolating ν-wise in the χ^2 table. These compare with exact values of $2{\cdot}22$ and $3{\cdot}11$ (Table 4).

If a control chart is set up for the mean of m ranges of n the degrees of freedom become $m\nu_n(\beta_2)$.

(*b*) If there is no information on which to base a correction for non-normality, what errors is non-normality likely to introduce? If β_2 is small, the upper limits are set too high and it will be difficult to detect lack of control; conversely, if it is found that few or no points fall above the limits, this suggests that the population β_2 may be low. If β_2 is large, too many ranges will fall above the limits; thus for the exponential population it was shown in § 5 that more than 5 % of ranges of four lie above the normal 1 % point. This is, however, rather an extreme case of non-normality. Conversely, if an excessive number of ranges fall above the control lines, it is possible that the ranges arise from sampling a stable but non-normal population and that the process is 'in control'. Indeed, it might require a lengthy analysis to decide by purely statistical means whether or not such a set of ranges are consistent with a state of control.

(iii) *Comparison of several variances.* In the first two applications, only one unknown population standard deviation is involved. Now consider the problem of testing the hypothesis that two populations have the same variance, given the mean ranges of random samples from the two populations. If we continue to use the χ^2 approximation to the distribution of range, we refer the range ratio

$$R = \frac{\overline{w}_1 d_{n_2}}{\overline{w}_2 d_{n_1}}$$

to the F tables with $(m_1\nu_1, m_2\nu_2)$ degrees of freedom. Here $\overline{w}_i$ is the mean of m_i ranges of n_i from the ith population and ν_i are the effective degrees of freedom for a range of n_i from a normal population (Cox, 1949).

Suppose now that the populations are not normal. Provided that the populations have approximately the same β_2, the bias factors $B_n(\beta_2)$ affect, for $n_i \leqslant 5$, numerator and denominator almost equally, even if $n_1 \neq n_2$. If the population β_2's were very different this would

not be true, but a test of the equality of the variances of populations of very different shapes is probably not a common problem, and this case will not be considered further. Therefore the only effect of non-normality is that the effective degrees of freedom should be

$$(m_1\nu_{n_1}(\beta_2), m_2\nu_{n_2}(\beta_2)), \quad \text{where} \quad \nu_n(\beta_2) = 2/C_n^2(\beta_2).$$

Table 7 gives some conclusions, based on § 5, for the case $n_1 = n_2 = 4$, $m_1 = m_2 = 1$, i.e. for the ratio of the ranges of single samples of four, where the population is (*a*) normal,

Table 7. *Percentage points of range ratio test for single samples of four*

		10 %	5 %	1 %
Normal population	Exact	2·4	3·1	5·6
	χ^2 approximation	2·2	2·8	4·5
	χ approximation	2·4	3·1	5·7
Exponential population*	Exact	3·3	4·7	9·6
	χ^2 approximation	3·5	5·1	11·0

* The χ approximation is difficult to compute in this case.

(*b*) exponential. The conclusions are first that while the χ approximation is very good indeed for the normal population, the χ^2 approximation is adequate in all cases, particularly when it is remembered that in nearly all applications the means of several ranges will be used, and not single ranges, and that the approximate percentage points will then be closer to the exact ones. Secondly, as would be expected, the use of the normal percentage points would be considerably in error with a population as non-normal as the exponential; the error would be much greater if the means of several ranges were used.

Box (1953) has shown the serious effect of non-normality on the conventional tests for comparing variances. His results are not, however, directly comparable with the present ones, because his are concerned with the comparison of single large samples. The best way of comparing the relative stability of tests based on the range and on the standard deviation, for numbers of small samples, is to compare:

(*a*) $C_n(\beta_2)/C_n(3)$, which shows how the coefficient of variation of an estimate based on the mean of ranges of n, changes relative to its value for $\beta_2 = 3$;

(*b*) the quantity $$\left\{\frac{(n-1)(\beta_2-3)}{2n}+1\right\}^{\frac{1}{2}}, \tag{10}$$

which by (7) is the corresponding quantity for the estimate of standard deviation based on a large number of mean squares of samples of n. This is done in Table 8. The conclusion is that comparative tests based on ranges of small samples are appreciably less affected by non-normality than the corresponding tests based on mean squares.

(iv) *Use of the range in the t-test.* In the preceding applications, the population standard deviation σ is of direct interest; we now consider a case in which σ is a nuisance parameter. Suppose that $\bar{x}$ is the mean of r observations from a population of mean μ and standard deviation σ and that σ is estimated from $\overline{w}_{m,n}$, the mean of m ranges of samples of n. To test the hypothesis that $\mu = 0$, the natural analogue of the t criterion is

$$u = \frac{\bar{x}d_n\sqrt{r}}{\overline{w}_{m,n}}. \tag{11}$$

Table 8. *Relative variation with β_2 of coefficient of variation of estimates of σ from (a) range, (b) root-mean-square*

n \ β_2	1·0	2·0	3·0	4·0	5·0	7·0	9·0
2 (*a*)	1·27	0·99	1·00	1·04	1·09	1·18	1·27
(*b*)	0·71	0·87	1·00	1·12	1·22	1·41	1·58
3 (*a*)	1·07	0·89	1·00	1·09	1·17	1·29	1·39
(*b*)	0·58	0·82	1·00	1·15	1·29	1·53	1·73
4 (*a*)	0·83	0·81	1·00	1·13	1·23	1·34	1·41
(*b*)	0·50	0·79	1·00	1·17	1·32	1·58	1·80
5 (*a*)	0·67	0·78	1·00	1·16	1·28	1·41	1·49
(*b*)	0·45	0·77	1·00	1·18	1·34	1·61	1·84

Table 9. *Percentage points of $\bar{x}d_n\sqrt{r}/\bar{w}_{m,n}$*

Size of sample n	No. of samples m	10% point			5% point			1% point		
		Exact	χ approx.	χ^2 approx.	Exact	χ approx.	χ^2 approx.	Exact	χ approx.	χ^2 approx.
5	1	2·02	2·02	2·04	2·63	2·64	2·68	4·38	4·39	4·51
	3	1·75	1·75	1·75	2·15	2·15	2·15	3·02	3·03	3·03
8	1	1·85	1·85	1·86	2·32	2·33	2·35	3·45	3·50	3·56
	3	1·71	1·71	1·71	2·07	2·07	2·08	2·83	2·83	2·85

The exact percentage points of u for a normal population have been found by numerical integration by Lord (1947), and approximations based on the χ approximation to the range distribution have been given by Patnaik (1950). The percentage points based on the χ^2 approximation to the range distribution are obtained by referring

$$u' = \frac{\bar{x}d_n\sqrt{r}}{\bar{w}_{m,n}(1-1/\nu)} \tag{12}$$

to the t table with $\nu' = \frac{1}{2}\nu = \frac{1}{2}m\nu_n$ degrees of freedom. The divisor $(1-1/\nu)$ simulates the bias of the estimate of standard deviation in the t-test, and the effective degrees of freedom, $m\nu_n$, of the mean of m ranges of n, are divided by four to represent the transition from a χ^2 variate, to the χ variate occurring in the denominator of t. Table 9 compares the exact percentage points of $\bar{x}d_n\sqrt{r}/\bar{w}_{m,n}$ with those obtained by the two methods of approximation.

The χ^2 approximation is less accurate than the χ approximation, although both are very good; the advantage of the χ^2 approximation is that its use requires only a table of ν_n, whereas the χ approximation requires a double table of scale factors and degrees of freedom as functions of m, n.

If the population is not normal it is natural to replace (12) by testing

$$\frac{\bar{x} d_n B_n(\beta_2) \sqrt{r}}{\overline{w}_{m,n}(1 - \nu_n^{-1}(\beta_2))}, \tag{13}$$

as t with $\frac{1}{4}m\nu_n(\beta_2)$ degrees of freedom. A more precise treatment would allow for the effect of correlation between the mean and the mean range; the general effect of this will be to increase the dispersion of (13) when β_2 is small and to decrease the dispersion when β_2 is high. Except in very small samples a change in the degrees of freedom is not important, and the only effect of non-normality is the insertion of the factor $B_n(\beta_2)$; this will not be important except in extreme cases and so as a general rule this test is not appreciably affected by non-normality.

The broad conclusions of § 6 are that if β_2 is known, it is possible to make a rough correction for non-normality in methods that use ranges of small samples, and that on the whole these methods are less affected by non-normality than corresponding methods using variances of small samples.

I am grateful to Mr D. A. East and Miss Patricia Johnson for their care with the numerical work.

REFERENCES

Box, G. E. P. (1953). *Biometrika*, **40**, 318.
Cox, D. R. (1949). *J.R. Statist. Soc.* B, **11**, 101.
Hartley, H. O. & David, H. A. (1954). *Ann. Math. Statist.* **25**, 85.
Jeffreys, H. & Jeffreys, B. S. (1946). *Methods of Mathematical Physics*. Cambridge University Press.
Kendall, M. G. (1943). *The Advanced Theory of Statistics*, **1**. London: Griffin and Co.
Link, R. F. (1950). *Ann. Math. Statist.* **21**, 112.
Lord, E. (1947). *Biometrika*, **34**, 41.
Maguire, B. A., Pearson, E. S. & Wynn, A. H. A. (1952). *Biometrika*, **39**, 168.
Patnaik, P. B. (1950). *Biometrika*, **37**, 78.
Pearson, E. S. (1950). *Biometrika*, **37**, 88.
Pearson, E. S. & Adyanthaya, N. K. (1929). *Biometrika*, **21**, 259.
Pearson, E. S. & Hartley, H. O. (1942). *Biometrika*, **32**, 301.
Pearson, E. S. & Hartley, H. O. (1951). *Biometrika*, **38**, 463.
Pearson, K. (1931). *Tables for Statisticians and Biometricians*, **2**. Cambridge University Press.
Plackett, R. L. (1947). *Biometrika*, **34**, 120.
Shone, K. J. (1949). *J.R. Statist. Soc.* B, **11**, 85.
Tippett, L. H. C. (1925). *Biometrika*, **17**, 364.

THE DISTRIBUTION OF THE RATIO, IN A SINGLE NORMAL SAMPLE, OF RANGE TO STANDARD DEVIATION

BY H. A. DAVID, H. O. HARTLEY AND E. S. PEARSON

Commonwealth Scientific and Industrial Research Organization, Sydney, Australia and University College, London

1. INTRODUCTION

A number of recent papers have discussed the use in statistical analysis of the 'Studentized' range, that is to say, the ratio of (*a*) the range in a sample of n observations from a population having standard deviation σ to (*b*) an *independent* root-mean-square estimate of σ based on ν degrees of freedom. Tables of the upper 5 and 1 % points of this ratio, in the case of normal variation, have been given by Pearson & Hartley (1954, Table 29). In the following paper we are concerned with a different statistic, namely, the ratio of the range, w, to the standard deviation, s, both measures of variation being calculated on the same sample of n observations. While the first ratio can play a useful part in providing certain rapid tests in the analysis of variance, the second, whose value depends only on the configuration of a particular sample, may be useful (with suitable tables) in detecting heterogeneity of data or departure from normality.

Our interest in this matter arose as a result of correspondence between one of us and Dr Joseph Berkson who has for some time carried out a routine scrutiny of data by making a comparison of the range and standard deviation estimators of σ; in this connexion he initiated an empirical investigation into the correlation between the estimators in order to determine the standard error of the difference between them. Some years earlier, however, G. A. Baker (1946) suggested the use of the ratio w/s as a means of detecting lack of homogeneity, and showed by an artificial sampling experiment that the distribution of this ratio might be expected to change considerably with the form of the parent population.

The object of the present paper is to provide tables of certain upper and lower percentage points of $u = w/s$ for samples of n observations from a single normal population. Two methods of attack will be used:

(*a*) We shall show that the exact moments of the distribution of u can be derived quite simply from the known moments of s and w. Hence approximations to the percentage points may be obtained from representing the distribution by any suitable frequency curve having the same moments, e.g. by a curve of the Pearson system.

(*b*) Using a method employed by Pearson & Chandra Sekar (1936), we shall show how in small samples values of the upper percentage points of u may be calculated exactly.

In an overlapping region of the table, these two methods provide some confirmation of the accuracy of the results.

Finally, we shall give some numerical illustrations of how the ratio might be used to provide a quick assessment of the homogeneity or normality of data.

2. METHOD USING THE MOMENTS OF THE DISTRIBUTION OF $u = w/s$

Let $x_1, x_2, \ldots, x_n$ represent a sample of n independent observations from a normal population having variance σ^2. Write

$$\bar{x} = \sum_{i=1}^{n} x_i/n, \quad s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1), \quad w = x_{\max.} - x_{\min.}.$$

It may be shown, following an argument similar to that used by Geary (1933, p. 185; 1936, p. 296) and since employed by Kamat (1953, p. 124), that the joint distribution of the ratios $(x_i - \bar{x})/s (i = 1, 2, \ldots, n)$ is independent of s. Hence any function of these ratios and in particular the difference of $(x_{\max.} - \bar{x})/s$ and $(x_{\min.} - \bar{x})/s$, or w/s, will be independent of s. From this it follows that

$$\mathscr{E}(u^r) = \mathscr{E}(w/s)^r = \mathscr{E}(w^r)/\mathscr{E}(s^r). \tag{1}$$

On this basis the moments of $u = w/s$ may be computed, making use of the moments of w as far as they have been determined, and the fully known moments of s. We shall be concerned here only with the moments of w up to the fourth; for $n \leqslant 20$ these have been given by Hartley & Pearson (1951) while for $n \geqslant 20$ values of the standard deviation and the β_1, β_2 coefficients were computed by Tippett (1925) (with some later modification by Pearson (1926)) at $n = 20, 60, 100, 200, 500$ and 1000. $\mathscr{E}(w)$ was, of course, given by Tippett for all sample sizes up to 1000.

The procedure was then as follows:

(*a*) Taking as framework the sample sizes $n = 10, 15, 20, 60, 100, 200, 500, 1000$, values of $\mathscr{E}(w^r)$ for $r = 2, 3, 4$ were found from the moments about the mean.

(*b*) The corresponding expectations of s^r were obtained from

$$\mathscr{E}(s^r) = \sigma^r (2/(n-1))^{\frac{1}{2}r}\, \Gamma\{\tfrac{1}{2}(n-1+r)\}/\Gamma\{\tfrac{1}{2}(n-1)\}, \tag{2}$$

and hence $\mathscr{E}(u^r)$ was determined from (1).

(*c*) The 2nd, 3rd and 4th moments of u about zero were then transferred to the mean, $\bar{u} = \mathscr{E}(u) = \mathscr{E}(w)/\mathscr{E}(s)$, and finally values of σ_u, $\beta_1(u)$ and $\beta_2(u)$ obtained. With the values for the moments of w at present available for large n, σ_u could only be calculated to three and the beta coefficients to two significant figures.

(*d*) Pearson curves with the same first four moments were then used to approximate to upper and lower $100\alpha\,\%$ points for $\alpha = 0{\cdot}10, 0{\cdot}05, 0{\cdot}025, 0{\cdot}01$ and $0{\cdot}005$ and for the framework values of n quoted in (*a*) above.

It was found that if these percentage points were expressed as standardized deviates, $U_\alpha = (u_\alpha - \bar{u})/\sigma_u$, then the resulting values of U_α were sufficiently smooth for interpolation in the panels of the frame for n. Using these interpolated values and the formula

$$u_\alpha = \bar{u} + U_\alpha \sigma_u,$$

where $\bar{u}$ is known to four and σ_u to three decimal places,* the panels were filled in to give additional percentage points for u at $n = 12, 30, 40, 50, 80$ and 150.

For $n \leqslant 20$ the percentage points should be accurate to within $\pm 0{\cdot}02$ and a comparison with the exact percentage points (see § 3, Table 1), where these are available, does not reveal discrepancies beyond $0{\cdot}01$. For $n > 20$ the uncertainty in the beta coefficients of w alone may result in an uncertainty of $\pm 0{\cdot}02$ in the percentage points.

* Apart from Tippett's (1925) values of σ_w required for σ_u, additional values at $n = 30, 45$ and 75 have been given by Pearson (1932, p. 405).

3. Derivation of the exact values of the upper percentage points in small samples

We have here followed a method previously used by Pearson & Chandra Sekar (1936) when determining upper percentage points of the distribution of $(x_{\max.}-\bar{x})/s$ in sampling from a normal population. This method proceeds as follows.

Suppose that the observations x_i $(i = 1, 2, \ldots, n)$ are arranged in some random order, say the order in which they are drawn. Then we may select a particular pair, say the jth and kth drawn, and form the ratio

$$u' = (x_j - x_k)/s, \tag{3}$$

where s is the sample standard deviation defined above. It is then easy to show that the distribution of u' is functionally related to that of Student's t. This follows because the total sum of squares of deviations from the sample mean, $(n-1)\,s^2$, contains the single degree of freedom component $\frac{1}{2}(x_j - x_k)^2$, so that

$$(n-1)\,s^2 = \tfrac{1}{2}(x_j - x_k)^2 + \chi^2\sigma^2,$$

χ^2 being based on $n-2$ degrees of freedom and being independent of $\frac{1}{2}(x_j - x_k)^2$. In fact, we can write in more detail

$$\chi^2\sigma^2 = \sum_{i \neq j,\, k} (x_i - \bar{x}')^2 + 2(n-2)\,(\bar{x}' - \bar{x}'')^2/n,$$

with $$\bar{x}' = \sum_{i \neq j,k} x_i/(n-2), \quad \bar{x}'' = \tfrac{1}{2}(x_j + x_k).$$

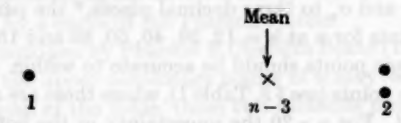
It follows that $$u' = \frac{(x_j - x_k)\,\sqrt{(n-1)}}{\{\chi^2\sigma^2 + \frac{1}{2}(x_j - x_k)^2\}^{\frac{1}{2}}} = \frac{t\,\sqrt{\{2(n-1)\}}}{\{t^2 + n - 2\}^{\frac{1}{2}}}, \tag{4}$$

where t is based on $n-2$ degrees of freedom. If now we consider a particular sample, there will be $n(n-1)$ different combinations of j and k, each providing a value of u'. These $n(n-1)$ values may be arranged in descending order of magnitude and denoted by

$$u'_{[1]} \geqslant u'_{[2]} \geqslant \ldots \geqslant u'_{[n(n-1)]}.$$

Clearly the distribution of u', determined from (4), for a pair j, k of x's taken at random will be formed from the sum of the $n(n-1)$ equally weighted distributions of $u'_{[1]}, \ldots, u'_{[n(n-1)]}$. The distribution of $u = w/s$ is the distribution of $u'_{[1]}$. In so far as there is no overlap of the distributions of $u'_{[1]}$ and $u'_{[2]}$ (and consequently of $u'_{[1]}$ and $u'_{[3]}$, etc.), the shape of the distribution of $u = u'_{[1]}$ will be that of the upper tail of the known distribution of u'.

As shown in the Appendix, the maximum value of $u'_{[2]}$ is $\{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$; this arises when the configuration of the sample is as follows:



For n of moderate size, such a configuration with $n-3$ observations having the same or very nearly the same value is most unlikely to occur, so that even when $u < \{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$ the overlap of the distributions of $u'_{[2]}$ and $u = u'_{[1]}$ may be of little importance.

Formally, we may say that for any positive quantity $U \geqslant \{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$,

$$\Pr\{u' \geqslant U\} = \Pr\{u' \text{ is } u_{[1]}\} \times \Pr\{u'_{[1]} = u \geqslant U\}$$

$$= \frac{1}{n(n-1)} \times \Pr\{u \geqslant U\}$$

or

$$\Pr\{u \geqslant U\} = n(n-1)\Pr\{u' \geqslant U\}. \tag{5}$$

This means that provided $U \geqslant \{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$, the upper tail area of the distribution of $u = w/s$ can be obtained exactly from that of u' which, through the relation (4), can be derived from the t-distribution. Alternatively, we can compute the $100\alpha\,\%$ point of u (say $u(\alpha, n)$) from the $100\alpha/\{n(n-1)\}\,\%$ point of t for $n-2$ degrees of freedom, using the formula

$$u^2(\alpha, n) = 2(n-1)\,t^2(\alpha', \nu)/\{\nu + t^2(\alpha', \nu)\}, \tag{6}$$

where $\alpha' = \alpha/\{n(n-1)\}$ and $\nu = n-2$.

The following table compares at $n = 10$, 15 and 20 the upper percentage points of $u = w/s$ obtained by this method and by that of the preceding section, for $\alpha = 0{\cdot}10$, $0{\cdot}05$, $0{\cdot}025$, $0{\cdot}01$ and $0{\cdot}005$. It also gives the limiting values $\{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$. It will be seen that at $n = 10$ four of the percentage points lie beyond the upper limit for $u'_{[2]}$, but that for $n = 15$ and 20 all the percentage points lie below this limit. Thus, strictly speaking, the second method of determining $u(\alpha, n)$ is not applicable for much of the table beyond $n = 10$. In fact, however, we find surprising agreement between the Pearson curve values and those derived from u' right up to $n = 20$, suggesting that the probability of $u'_{[2]}$ falling in the overlapping upper tail area of its distribution must be very small. As would be expected, the difference between the two results, where it appears, makes the limit obtained from the u'-distribution slightly larger than that based on moments.

Table 1. *Comparison of percentage points for $u = w/s$ obtained by different methods*

n	Method of derivation	Percentage points, $u(\alpha, n)$					Max. value of $u'_{[2]}$
		$\alpha = 0{\cdot}10$	$0{\cdot}05$	$0{\cdot}025$	$0{\cdot}01$	$0{\cdot}005$	
10	From curve fitted by moments	3·57	3·69	3·78	3·88	3·94	—
	From u'-distribution	3·574	3·685⁻	3·777	3·875⁻	3·935⁻	3·674
15	From curve fitted by moments	4·02	4·17	4·29	4·43	4·52	—
	From u'-distribution	4·034	4·173	4·295⁺	4·435⁺	4·527	4·583
20	From curve fitted by moments	4·32	4·49	4·63	4·79	4·91	—
	From u'-distribution	4·342	4·496	4·635⁺	4·799	4·910	5·339

4. COMPOSITION OF THE FINAL TABLE

The results following from the methods of the two preceding sections have been put together in Table 6 printed at the end of the paper (p. 491 below).

For the lower percentage points, the lines corresponding to $n = 10$, 15, 20, 60, 100, 200, 500 and 1000 have been computed by the method of § 2, i.e. from the fitted Pearson curves,

and the remaining lines by interpolation. The same applies to the upper percentage points for $n \geqslant 20$. For $n < 20$, the values shown to three decimal accuracy are exact, being in every case less than the limit $\{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$, and have been computed from formula (6). For the remaining upper percentage points with $n \leqslant 20$, the values yielded by formula (6) were reduced slightly, the reduction amounting to either 1 or 2 units in the second decimal. This reduction was assessed by smoothing the discrepancies between the two methods shown in Table 1.

We have not attempted to interpolate for $n > 200$. A rough interpolation can easily be made by graphical or other methods, but if more accurate values of percentage points of w/s for such large sample sizes were really required, it would seem necessary to compute additional values of the higher moments of the range between Tippett's values at $n = 200$, 500 and 1000.

5. Illustrations of possible uses of the table of percentage points

We have pressed forward with the preparation of this table, involving certain inaccuracy for the larger samples, because the practical usefulness of the ratio w/s can only be tested with a table available. The following examples illustrate some of a variety of ways in which the table might be used, but it must be emphasized that no claim is made that the test is better than other simple tests which have been or could be devised.

(5·1) *Testing for lack of homogeneity of data*

As mentioned above Dr Joseph Berkson has made it a practice in course of consultation work with laboratory colleagues, to compare standard deviations derived from samples of 50–200 observations with the range estimate of σ found by dividing the sample range by factor d_n obtained from Tippett's (1925) table. He writes:

'On occasion there was a considerable difference between the range and mean-square estimates and in these instances, almost invariably, further investigation disclosed that either there had been an arithmetic error, or the observations were very different from normally distributed; in particular there were one or more observations outlying from the rest. This sort of experience, accumulated over a long period of time, built up the impression that the range estimate of the standard deviation of a normal distribution was a close equivalent of the mean-square estimate.'

To illustrate the use of our table in this connexion we have considered some of the frequency distributions of physical characters published by Mahalanobis, Majumdar & Rao (1949) in the report on an anthropometric survey of the United Provinces of India undertaken in 1941. From these data we have taken the frequency distributions of head length given for each of 23 racial groups. (Appendix 7 of the paper and also Tables 2·1 and 6·1.) The sample size varies from 57 to 197; for each of the 23 groups we have calculated the ratio w/s and the results are plotted in Fig. 1 on which are also drawn the upper and lower 5 and 1% significance levels for the ratio. For these levels, smooth curves have been drawn through points taken from Table 6. It will be seen that for five of the twenty-three groups w/s falls beyond the upper 5% level and for two of these, well beyond the 1% level. On looking at the original frequency distributions, it was found that in each of these five cases there was one or more than one very divergent measurement, always in the sense of a small head length. Table 2 gives some idea of the position.

The notation x_I has been used to denote the lowest observation in the sample and x_{II} the lowest but one, except in Group 12 where it represents the lowest but two. The ratios $(x_I - \bar{x})/s$ and $(x_I - x_{II})/s$ give some idea of the divergence of the outliers.

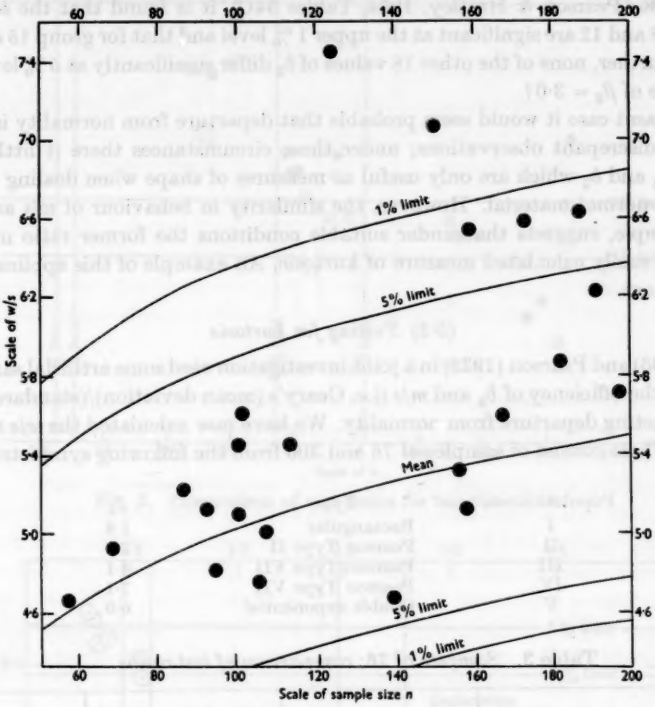


Fig. 1 Head length data for 23 Indian racial groups.

Table 2. *Data from five groups showing abnormal w/s (the measurements of head length are in millimetres)*

Group no.	n	$\bar{x}$	s	w	w/s	$(x_I - \bar{x})/s$	$(x_I - x_{II})/s$	b_2
6	150	186·10	6·94	49	7·1	−5·2	−2·0	6·82
7	124	186·94	6·04	45	7·5	−4·6	−2·6	5·61
8	187	181·87	5·88	39	6·6	−4·1	−1·2	4·30
12	173	187·69	6·69	44	6·6	−3·5 (2)*	−1·2	4·46
15	159	186·85	6·42	42	6·5	−3·7	−1·4	3·72

* Two observations fell at the extreme value of 164 mm.

Without inside knowledge of the investigation we cannot, of course, judge whether any of these outlying readings are due to faulty recording, but had a test of this character been applied to the distributions originally there would have been a *prima facie* case for further scrutiny of the data before, for example, calculating higher moments.

The Indian Report gives also the values of the shape coefficients $\sqrt{b_1}$ and b_2 for each of the twenty-three distributions. Taking b_2 as a measure of kurtosis most likely to be related to w/s and referring to a table of significance levels to test for departure from normality (see Pearson, 1930; Pearson & Hartley, 1954, Tables 34C), it is found that the b_2 values for groups 6, 7, 8 and 12 are significant at the upper 1 % level and that for group 15 at the upper 5 % level. Further, none of the other 18 values of b_2 differ significantly at 5 % level from the normal value of $\beta_2 = 3{\cdot}0$!

In the present case it would seem probable that departure from normality is due to the presence of discrepant observations; under these circumstances there it little reason to calculate $\sqrt{b_1}$ and b_2 which are only useful as measures of shape when dealing with homogeneous non-normal material. However, the similarity in behaviour of w/s and b_2 in the present example, suggests that under suitable conditions the former ratio may provide a useful and easily calculated measure of kurtosis. An example of this application will be considered next.

(5·2) *Testing for kurtosis*

Geary (1935) and Pearson (1935) in a joint investigation used some artificial sampling data to compare the efficiency of b_2 and m/s (i.e. Geary's (mean deviation)/(standard deviation) ratio) in detecting departure from normality. We have now calculated the w/s ratio on the same data. These consist of samples of 76 and 300 from the following symmetrical populations:

Population no.	Type	β_2
I	Rectangular	1·8
II	Pearson Type II	2·5
III	Pearson Type VII	4·1
IV	Pearson Type VII	7·1
V	Double exponential	6·0

Table 3. *Samples of* 76; *comparison of test ratios*

Population ...	I (rectangular)			IV (Type VII, $\beta_2 = 7{\cdot}1$)			V (double exponential)		
Test ratio used ...	m/s	b_2	w/s	m/s	b_2	w/s	m/s	b_2	w/s
Number of sample points:									
Within 5 % limits	0	0	0	3	2	7	1	1	3
Between 5 and 1 % limits	2	0	0	1	4	1	1	5	3
Beyond 1 % limits	8	10	10	6	4	2	8	4	4

A comparison of the b_2 and w/s values for ten samples with $n = 76$ from populations nos. I, IV and V and for six samples with $n = 300$ from populations II and III is made in Figs. 2 and 3. Similar diagrammatic comparisons between b_2 and Geary's ratio m/s were given in Pearson's (1935), Figs. 3 and 4. The high correlation between b_2 and w/s will be noted, particularly for the platykurtic distributions I and II. The relative power of the three ratios in testing for departure from normality is summarized in Tables 3 and 4 as far as these very limited sampling results are concerned. It will be seen that the w/s ratio is in some cases as effective as the other two statistics, but is never more effective than b_2. On

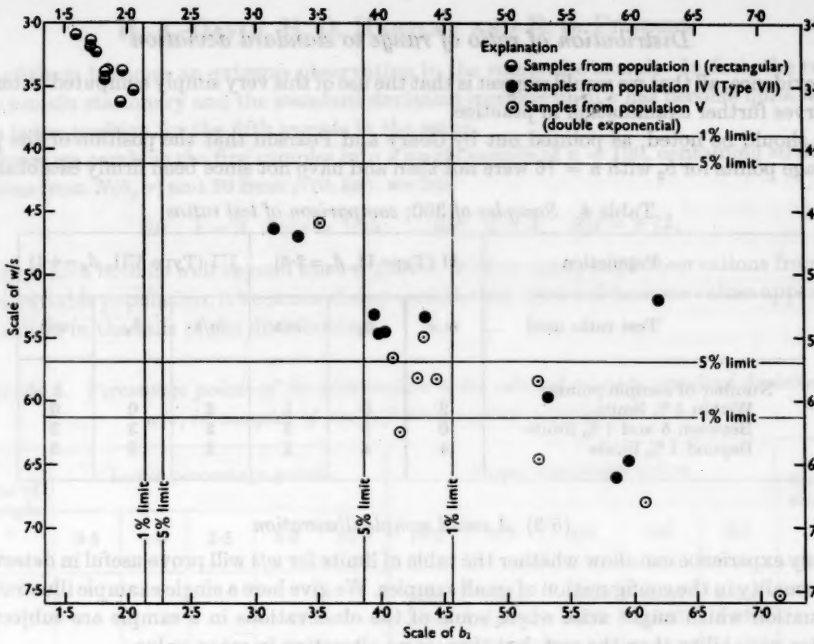


Fig. 2. Comparison of test ratios for samples of 76.

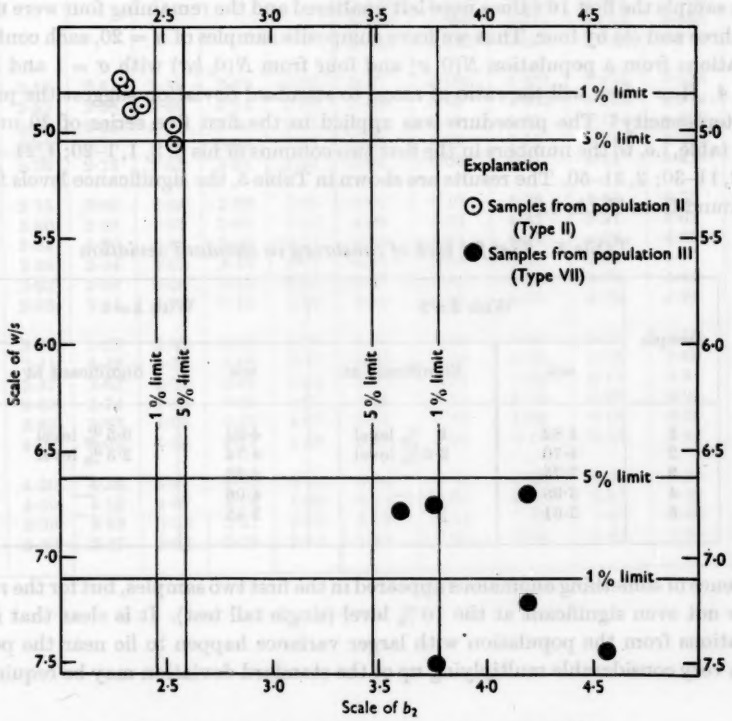


Fig. 3. Comparison of test ratios for samples of 300.

this evidence, all that we would suggest is that the use of this very simply computed criterion deserves further examination in practice.

It should be noted, as pointed out by Geary and Pearson that the position of the percentage points for b_2 with $n = 76$ were not then and have not since been firmly established.

Table 4. *Samples of* 300; *comparison of test ratios*

Population ...	II (Type II, $\beta_2 = 2{\cdot}5$)			III (Type VII, $\beta_2 = 4{\cdot}1$)		
Test ratio used ...	m/s	b_2	w/s	m/s	b_2	w/s
Number of sample points:						
Within 5 % limits	2	0	1	2	0	0
Between 5 and 1 % limits	0	2	3	2	3	3
Beyond 1 % limits	4	4	2	2	3	3

(5·3) *A small sample illustration*

Only experience can show whether the table of limits for w/s will prove useful in detecting abnormality in the configuration of small samples. We give here a single example illustrating a situation which might arise where some of the observations in a sample are subject to greater variability than the rest, but there is no alteration in mean value.

Samples of 20 values were taken from Wold's (1948) *Table of Random Normal Deviates*. In each sample the first 16 values were left unaltered and the remaining four were multiplied (*a*) by three and (*b*) by four. Thus we have composite samples of $n = 20$, each containing 16 observations from a population $N(0, \sigma)$ and four from $N(0, k\sigma)$ with $\sigma = 1$ and (*a*) $k = 3$, (*b*) $k = 4$. How often will the ratio of range to standard deviation suggest the presence of this heterogeneity? The procedure was applied to the first five series of 20 numbers in Wold's table, i.e. to the numbers in the first two columns of his p. 2, 1, 1–20; 1, 21–40; 1, 41–2, 10; 2, 11–30; 2, 31–50. The results are shown in Table 5, the significance levels for $n = 20$ being found from Table 6.

Table 5. *Test for lack of constancy in standard deviation*

Sample	With $k = 3$		With $k = 4$	
	w/s	Significant at	w/s	Significant at
1	4·82	1 % level	4·93	0·5 % level
2	4·70	2·5 % level	4·74	2·5 % level
3	3·75	—	4·26	—
4	3·98	—	4·06	—
5	3·61	—	3·45	—

Evidence of something anomalous appeared in the first two samples, but for the remainder w/s was not even significant at the 10 % level (single tail test). It is clear that if all four observations from the population with larger variance happen to lie near the population mean, a very considerable multiplying up of the standard deviation may be required before

any of them becomes an extreme observation in the composite sample. In fact, the range may remain stationary and the standard deviation increase until k has become quite large. This is the position for the fifth sample in the series.

If now we combine the five samples into a single sample of $n = 100$, containing 80 observations from $N(0, \sigma)$ and 20 from $N(0, k\sigma)$, we find

$$\text{for} \quad k = 3, \quad w/s = 7{\cdot}95; \qquad \text{for} \quad k = 4, \quad w/s = 8{\cdot}72,$$

results which both lie well beyond the 0·5% level. With as many as 20 observations from the more variable population, it becomes almost certain that there will be some values appearing as outliers in the tails of the distribution.

Table 6. *Percentage points of the distribution of the ratio of range to standard deviation, w/s, in samples of size n from a normal population*

Size of sample n	Lower percentage points					Upper percentage points					Size of sample n
	0·5	1·0	2·5	5·0	10·0	10·0	5·0	2·5	1·0	0·5	
						1·997	1·999	2·000	2·000	2·000	3
						2·409	2·429	2·439	2·445	2·447	4
						2·712	2·753	2·782	2·803	2·813	5
						2·949	3·012	3·056	3·095	3·115	6
						3·143	3·222	3·282	3·338	3·369	7
						3·308	3·399	3·471	3·543	3·585	8
						3·449	3·552	3·634	3·720	3·772	9
10	2·47	2·51	2·59	2·67	2·77	3·57	3·685	3·777	3·875	3·935	10
11	2·53	2·58	2·66	2·74	2·84	3·68	3·80	3·903	4·012	4·079	11
12	2·59	2·65	2·73	2·80	2·91	3·78	3·91	4·01	4·134	4·208	12
13	2·65	2·70	2·78	2·86	2·97	3·87	4·00	4·11	4·244	4·325	13
14	2·70	2·75	2·83	2·91	3·02	3·95	4·09	4·21	4·34	4·431	14
15	2·75	2·80	2·88	2·96	3·07	4·02	4·17	4·29	4·43	4·53	15
16	2·80	2·85	2·93	3·01	3·13	4·09	4·24	4·37	4·51	4·62	16
17	2·84	2·90	2·98	3·06	3·17	4·15	4·31	4·44	4·59	4·69	17
18	2·88	2·94	3·02	3·10	3·21	4·21	4·38	4·51	4·66	4·77	18
19	2·92	2·98	3·06	3·14	3·25	4·27	4·43	4·57	4·73	4·84	19
20	2·95	3·01	3·10	3·18	3·29	4·32	4·49	4·63	4·79	4·91	20
30	3·22	3·27	3·37	3·46	3·58	4·70	4·89	5·06	5·25	5·39	30
40	3·41	3·46	3·57	3·66	3·79	4·96	5·15	5·34	5·54	5·69	40
50	3·57	3·61	3·72	3·82	3·94	5·15	5·35	5·54	5·77	5·91	50
60	3·69	3·74	3·85	3·95	4·07	5·29	5·50	5·70	5·93	6·09	60
80	3·88	3·93	4·05	4·15	4·27	5·51	5·73	5·93	6·18	6·35	80
100	4·02	4·09	4·20	4·31	4·44	5·68	5·90	6·11	6·36	6·54	100
150	4·30	4·36	4·47	4·59	4·72	5·96	6·18	6·39	6·64	6·84	150
200	4·50	4·56	4·67	4·78	4·90	6·15	6·38	6·59	6·85	7·03	200
500	5·06	5·13	5·25	5·37	5·49	6·72	6·94	7·15	7·42	7·60	500
1000	5·50	5·57	5·68	5·79	5·92	7·11	7·33	7·54	7·80	7·99	1000

APPENDIX

Determination of the maximum value of $u'_{[2]}$

Let x_i $(i = 1, 2, \ldots, n)$ denote the values of a random sample of n observations arranged in ascending order of magnitude, so that x_1 is the smallest and x_n the largest.* We wish to determine the maximum value of the ratio $u'_{[2]} = (x_{n-1} - x_1)/s$, where $s^2 = \sum_i (x_i - \bar{x})^2/(n-1)$. Since the value will not depend on the origin or scale of the x_i we may assume, without loss of generality, that $x_1 = 0$ and $x_{n-1} = 1$, so that $x_{n-1} - x_1 = 1$ and the maximum of $u'_{[2]}$ can be found by minimizing the sum of squares

$$S = (n-1)\,s^2 = \Sigma x_i^2 - (\Sigma x_i)^2/n.$$

This minimum will be found in three stages:

(1) With $x_1 = 0, x_{n-1} = 1$ and x_n held fixed, S is minimized if all the x_i for $i = 2, 3, \ldots, n-2$ have the same value, say X. For if $X = \sum_{i=2}^{n-2} x_i/(n-3)$, we may write

$$S = 1 + x_n^2 + \sum_{i=2}^{n-2} (x_i - X)^2 + (n-3)\,X^2 - (1 + x_n + (n-3)\,X)^2/n,$$

which is clearly minimized if all $x_i = X$ $(i = 2, 3, \ldots, n-2)$.

(2) Confining ourselves to samples in which $x_1 = 0$, $x_i = X$ $(i = 2, 3, \ldots, n-2)$ and $x_{n-1} = 1$, we may next show that S is minimized if $x_n = 1$. We now have

$$S = 1 + x_n^2 + (n-3)\,X^2 - (1 + x_n + (n-3)\,X)^2/n.$$

Then
$$\frac{\partial S}{\partial x_n} = 2(x_n - \bar{x}) > 0,$$

so that the smallest possible value of x_n, namely, $x_n = x_{n-1} = 1$, will minimize S.

(3) Next, confining ourselves to samples with

$$x_1 = 0, \quad x_n = x_{n-1} = 1 \quad \text{and} \quad x_i = X \quad (i = 2, 3, \ldots, n-2),$$

we must find the value of X which minimizes S. Clearly now

$$S = 2 + (n-3)\,X^2 - (2 + (n-3)\,X)^2/n,$$

an expression which is easily found to have a minimum at $X = \frac{2}{3}$. For this we have

$$S = 2 + \tfrac{4}{9}(n-3) - (2 + (n-3)\tfrac{2}{3})^2/n = \tfrac{2}{3}.$$

Hence
$$s^2 = S/(n-1) = 2/\{3(n-1)\},$$

and the maximum value of $u'_{[2]}$ is $\{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$.

For this configuration maximizing $u'_{[2]}$ we have a single observation at one extreme of the sample range, two equal observations at the other extreme and the remaining $n-3$ observations together at the mean of the whole, i.e. at a distance two-thirds of the way between the first extreme and the second extreme.

* Note the change from § 3, where the observations were taken in random order.

REFERENCES

BAKER, G. A. (1946). *Ann. Math. Statist.* **17**, 366.

GEARY, R. C. (1933). *Biometrika*, **25**, 184.

GEARY, R. C. (1935). *Biometrika*, **27**, 310, 353.

GEARY, R. C. (1936). *Biometrika*, **28**, 295.

HARTLEY, H. O. & PEARSON, E. S. (1951). *Biometrika*, **38**, 463.

KAMAT, A. R. (1953). *Biometrika*, **40**, 116.

MAHALANOBIS, P. C., MAJUMDAR, D. N. & RAO, C. R. (1949). *Sankhyā*, **9**, 89.

PEARSON, E. S. (1926). *Biometrika*, **18**, 173.

PEARSON, E. S. (1930). *Biometrika*, **22**, 239.

PEARSON, E. S. (1932). *Biometrika*, **24**, 404.

PEARSON, E. S. (1935). *Biometrika*, **27**, 333.

PEARSON, E. S. & CHANDRA SEKAR, C. (1936). *Biometrika*, **28**, 308.

PEARSON, E. S. & HARTLEY, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge University Press.

TIPPETT, L. H. C. (1925). *Biometrika*, **17**, 364.

WOLD, H. (1948). *Random Normal Deviates*. Tracts for Computers, no. 25. Cambridge University Press.

SAMPLING WITH CONTROL VARIABLES

By E. C. FIELLER and H. O. HARTLEY

1. The use of control variables in certain types of sample surveys is a well-known procedure. For example, if we wish to estimate the population mean η of a variable y, and can observe concomitantly a 'control variable' x for which the population mean ξ is known exactly, then with the help of the sample estimate of the regression of y on x we may estimate η as $\bar{y}+b(\xi-\bar{x})$. If x and y are highly correlated, this adjusted estimate will usually have considerably higher precision than the simple estimate $\bar{y}$ of η. Methods of sample survey employ numerous alternative procedures for utilizing a 'control variable' x correlated with the variable y for which estimates are required; we may mention here the well-known devices of 'ratio estimates' y/x, and that of a 'post-stratification' of y with regard to x.

2. In the study of frequency distributions by experimental sampling, we aim at estimating a set of relative frequencies in an unknown population from the observed proportions in a sample. It is well known that very large samples, N, are required in such a procedure, since the standard deviations of the estimates normally decrease only as $1/\sqrt{N}$. The advent of high-speed electronic calculators has, however, opened up the possibility of tabulating mathematical distributions to low decimal accuracy with the help of these methods, which are nowadays often referred to as 'Monte Carlo Calculations'. Nevertheless, the expense and effort of such calculations make it imperative to seek for methods of improving the low precision that they normally yield. It is natural, therefore, to ask whether the estimation of the theoretical frequency distribution of a variable y by sampling cannot be improved with the help of a control variable x, the true distribution of which has been previously determined.

In this note we discuss the use of such control variables in this way, and show that the resulting gain in precision may well be considerable. Indeed, it will be seen that with the help of the technique discussed, the value of the sample, as judged by the precision of the estimates derived from it, may often be increased three to fivefold at the cost of relatively little extra calculation.

The problems that we consider presented themselves to us, and are discussed in the following paragraphs, from the point of view of the empirical study of sampling distributions. It will be apparent, however, that the results at which we arrive may, in suitable circumstances, be applied in any sampling survey, the objective of which is to study the distribution of some characteristic in the population sampled. In this connexion the method is related to the device known as 'Double Sampling' (Neyman, 1938).

3. By way of illustration, we consider the sampling experiment summarized* in Table 1. $N = 1000$ samples of 5 $(X_1, X_2, \ldots, X_5)$ were drawn, with the help of Hollerith equipment, from a normal parent population with unit standard deviation (Wold, 1948). Table 1 gives (at rather wide intervals) the joint distribution in these samples of the two statistics

$$y = \text{the sample range} = X_{\max.} - X_{\min.}$$

and

$$x = \text{the sum of squared deviations} = \Sigma(X_i - \bar{X})^2.$$

* The fractional group frequencies for $x = \chi^2$ have arisen from the use of percentage points of χ^2 for the group end-points.

The expected marginal frequency distributions were calculated from tables of the χ^2 probability integral and tables of the probability integral of the range (Pearson & Hartley, 1954, Table 23), respectively. In both cases the agreement of observation with theory is extremely good, the respective values of χ^2 for goodness of fit being 3·64 for y and 8·46 for x, each for 5 degrees of freedom.

Let us now assume, for the sake of illustration, that the theoretical distribution of the range y is not known, and that we wish to estimate from the data in Table 1 the relative frequencies with which y falls into the six intervals 0 to 1, 1 to 2, etc., used in the compilation of Table 1. The observed marginal proportional frequencies 0·0560, 0·3305, ... are of course themselves one set of estimates, whose precision is given by the familiar formula for the binomial variance; but the question arises, whether we can utilize our exact knowledge of the expected x-frequencies to obtain other estimates of the y-frequencies that are of higher precision.

Table 1. *Joint distribution of range* (y) *and reduced sum of squares* (x) *in* 1000 *samples of* 5

Group end-points for range (y)	Group end-points for sum of squares (x)						Total frequencies for y		
	0	0·71	2·75	4·88	9·49	14·86 ∞	Observed	Expected	$N\hat{p}_{i.}$
0	49·5	6·5					56·0	45·0	51·8
1	5·0	305·0	20·5				330·5	336·6	327·9
2		39·0	306·0	69·0			414·0	407·5	396·0
3			2·5	142·0	18·5		163·0	173·2	184·8
4				10·0	19·5	4·0	33·5	34·0	37·3
5						3·0	3·0	3·7	2·1
∞									
Totals									
Observed	54·5	350·5	329·0	221·0	38·0	7·0	1000·0		(999·9)
Expected	50·0	350·0	300·0	250·0	45·0	5·0		1000·0	

Let us denote by n_{ij} the frequency with which, in a sample of N, the observed (y, x) pairs fall in the ith y-category and the jth x-category ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$; $\sum_{ij} n_{ij} = N$), and write the marginal frequencies as $n_{.j} = \sum_i n_{ij}$, $n_{i.} = \sum_j n_{ij}$. Likewise, let us denote by p_{ij} the unknown probability that the pair (y, x) falls in the i, jth cell, and write $p_{.j} = \sum_i p_{ij}$, $p_{i.} = \sum_j p_{ij}$ for the marginal probabilities. Now the product $p_{.j}(n_{ij}/n_{.j})$ is clearly an unbiased estimate of the cell probability p_{ij} involving the known $p_{.j}$ and an estimate $n_{ij}/n_{.j}$ of the conditional probability $p_{ij}/p_{.j}$. The $n_{ij}/n_{.j}$ are in fact maximum-likelihood estimators of the $p_{ij}/p_{.j}$ for those columns for which $n_{.j} > 0$. No such estimators of the $p_{ij}/p_{.j}$ are available for those columns for which $n_{.j} = 0$. If in these latter cases we use the marginal ratio $n_{i.}/N$ as an estimate of $p_{ij}/p_{.j}$ we reach as an estimate of $p_{i.}$ the sum of products

$$\hat{p}_{i.} = \sum p_{.j} u_{ij}, \quad \text{where} \quad u_{ij} = \begin{cases} n_{ij}/n_{.j} & \text{if} \quad n_{.j} > 0, \\ n_{i.}/N & \text{if} \quad n_{.j} = 0. \end{cases} \tag{1}$$

In the case of our example, the 'revised' values of the y-frequencies derived from formula (1) are given in the final column of Table 1. These revised values do not appear, at first sight, to improve greatly on the original estimates $Np_{i.}$; if they are formally compared with their expectations by χ^2, we obtain 3·37 against 3·64. The justification for preferring the revised estimates $N\hat{p}_{i.}$ is provided, however, by an examination of their statistical properties. We note in the first place that

$$\sum_{i=1}^{m} \hat{p}_{i.} = 1 \tag{2}$$

and

$$\mathscr{E}(\hat{p}_{i.}) = p_{i.} - \sum_j q_{.j}^{N-1}(p_{ij} - p_{i.}p_{.j}), \tag{3}$$

where

$$q_{.j} = 1 - p_{.j}.$$

The first property (2) is obvious from (1), and the second (3) follows from an evaluation of $\mathscr{E}(u_{ij})$. We have

$$\begin{aligned}\mathscr{E}(u_{ij}) &= \Pr.\{n_{.j} > 0\}\, p_{ij}/p_{.j} + \Pr.\{n_{.j} = 0\}\,(p_{i.} - p_{ij})/(1 - p_{.j}) \\ &= (1 - q_{.j}^N)\, p_{ij}/p_{.j} + q_{.j}^N (p_{i.} - p_{ij})/(1 - p_{.j}) \\ &= p_{ij}/p_{.j} + q_{.j}^{N-1}(p_{i.} - p_{ij}/p_{.j}),\end{aligned} \tag{4}$$

and (3) follows from a combination of (1) and (4). The bias of $\hat{p}_{i.}$, given by $\sum_j q_{.j}^{N-1}(p_{ij} - p_{i.}p_{.j})$ is of the order of magnitude of $\sum_j e^{-Np_{.j}}$, and is clearly negligible if it is arranged that all expected x-frequencies $Np_{.j}$ are, say, $\geqslant 8$. This can be done in advance of sampling, since the $p_{.j}$ are known.

4. We turn now to the variance of $\hat{p}_{i.}$, and to this end require the variances and covariances of the u_{ij}. We have

$$\begin{aligned}\mathscr{E}(u_{ij}^2) &= \mathscr{E}(u_{ij}^2 \mid n_{.j} > 0).(1 - q_{.j}^N) + \mathscr{E}(u_{ij}^2 \mid n_{.j} = 0).q_{.j}^N \\ &= \quad A \quad + \quad B \quad \text{(say)}.\end{aligned} \tag{5}$$

Now since for any fixed $n_{.j} > 0$ we have

$$\mathscr{E}(u_{ij}^2 \mid n_{.j}) = \frac{p_{ij}}{p_{.j}}\left(1 - \frac{p_{ij}}{p_{.j}}\right)\frac{1}{n_{.j}} + \left(\frac{p_{ij}}{p_{.j}}\right)^2, \tag{6}$$

it follows that

$$A = E'\frac{p_{ij}}{p_{.j}}\left(1 - \frac{p_{ij}}{p_{.j}}\right) + (1 - q_{.j}^N)\left(\frac{p_{ij}}{p_{.j}}\right)^2, \tag{7}$$

where

$$E' = E'(Np_{.j}, N) = \sum_{n_{.j}=1}^{N} \frac{1}{n_{.j}}\binom{N}{n_{.j}} p_{.j}^{n_{.j}} q_{.j}^{N - n_{.j}}. \tag{8}$$

To evaluate B, we note that if we are given that $n_{.j} = 0$, we have

$$u_{ij} = n_{i.}/N = \sum_{j' \neq j} n_{ij'}/N. \tag{9}$$

Using the variances and covariances of the n_{ij}, we find

$$\mathscr{E}(u_{ij}^2 \mid n_{.j} = 0) = (p_{i.} - p_{ij})^2/q_{.j}^2 + O(N^{-1}), \tag{10}$$

so that

$$\mathscr{E}(u_{ij}^2) = E'\frac{p_{ij}}{p_{.j}}\left(1 - \frac{p_{ij}}{p_{.j}}\right) + (1 - q_{.j}^N)\left(\frac{p_{ij}}{p_{.j}}\right)^2 + q_{.j}^{N-2}(p_{i.} - p_{ij})^2 + O(N^{-1}q_{.j}^N). \tag{11}$$

Applying the correction for mean from (4), we obtain, after some rearrangement of terms,

$$\text{var}\,(u_{ij}) = E'\frac{p_{ij}}{p_{.j}}\left(1 - \frac{p_{ij}}{p_{.j}}\right) + q_{.j}^{N-2}\left(p_{i.} - \frac{p_{ij}}{p_{.j}}\right)^2 + O(N^{-1}q_{.j}^N) + O(q_{.j}^{2N}). \tag{12}$$

By a similar procedure it can be shown that the covariances of u_{ij_1} and u_{ij_2} are of the order $O(q^N_{.j_1}, q^N_{.j_2})$, and can therefore be neglected.

To evaluate var $(\hat{p}_{i.})$, we now need an approximation to the quantity E' defined in equation (8). This quantity, the expectation of the reciprocal of a binomial variate excluding the zero class, is of importance in many aspects of sampling survey theory, and it has seemed worth while to study it in detail in an Appendix to this paper. For our present purpose, it is sufficient to use the approximation provided by the expansion (9) of the Appendix, namely,

$$E' = 1/Np_{.j} + q_{.j}/(Np_{.j})^2 + O\{(Np_{.j})^{-3}\}. \tag{13}$$

Using the first two terms in (12) and (13), and substituting (13) in (12), we obtain

$$\begin{aligned} \text{var}\,(\hat{p}_{i.}) &\simeq \sum_j \{p^2_{.j}\,\text{var}\,(u_{ij})\} \\ &= p_{i.}(1-p_{i.})/N - \sum_j (p_{ij}-p_{i.}p_{.j})^2/Np_{.j} \\ &\quad + \sum_j p_{ij}q_{.j}(1-p_{ij}/p_{.j})/N^2p_{.j} + \sum_j q^{N-2}_{.j}(p_{ij}-p_{i.}p_{.j})^2 \\ &= \text{I} + \text{II} + \text{III} + \text{IV}, \quad \text{say.} \end{aligned} \tag{14}$$

In comparison with I and II, which are each of order $1/N$, the terms III and IV in (14) may be ignored. The leading term I is the binomial variance of the unadjusted estimate $p_{i.}$, and the second, II, representing the gain in precision due to the use of $\hat{p}_{i.}$ in place of $p_{i.}$, may conveniently be termed the 'variance reduction'.

This latter term is zero if $p_{ij} = p_{i.}p_{.j}$, i.e. if x and y are independent, so that no gain in precision is achieved in this case. If, on the other hand, there is 'perfect dependence' between y and x, i.e. if there is a one-to-one correspondence of i and j such that $p_{i.} = p_{ij} = p_{.j}$ and $p_{ij} = 0$ for all other combinations of i and j, then var $(\hat{p}_{i.})$ vanishes. In general, the reduction term II will lie between zero and I, and if x and y are highly correlated there will usually be a considerable reduction in var $(\hat{p}_{i.})$ compared with the binomial variance I.

5. Table 2 shows the extent to which, in the sampling experiment summarized in Table 1, the process of adjustment has increased the reliability of our estimates of the proportionate frequencies in the distribution of y. The relative efficiency of the $\hat{p}_{i.}$, calculated as var $(p_{i.})$/var $(\hat{p}_{i.})$, ranges in the six groups from 150 to 480 %. It will be seen from the last two columns in Table 2 that the error in the adjusted estimates $\hat{p}_{i.}$ is about twice the standard error of $\hat{p}_{i.}$ in the first frequency group, and of the order of the standard error in the remaining five groups.

Table 2. *Relative efficiencies of the estimates given in Table* 1

Group end-points for range (y)	Estimated variance		Relative efficiency	Estimated standard error of $\hat{p}_{i.}$	Difference $\hat{p}_{i.}$ − true $p_{i.}$
	var $(p_{i.})$	var $(\hat{p}_{i.})$			
0	0·053	0·011	4·8	0·0033	0·0068
1	0·221	0·063	3·5	0·0079	−0·0087
2	0·243	0·104	2·3	0·0102	−0·0115
3	0·136	0·063	2·2	0·0079	0·0116
4	0·032	0·021	1·6	0·0046	0·0033
5	0·003	0·002	1·5	0·0014	−0·0016
∞					

6. The 'variance reduction', $\text{var}(p_{i.}) - \text{var}(\hat{p}_{i.})$, depending as it were on the strength of the association between the primary variable y and the control variable x, can be related to the classical measure of 'mean-square contingency' introduced by Karl Pearson. In fact it follows from the first two terms in (14) that the proportional variance reduction for the ith y-group is

$$1 - \text{var}(\hat{p}_{i.})/\text{var}(p_{i.}) = \sum_j \frac{(p_{ij} - p_{i.}p_{.j})^2}{p_{i.}p_{.j}(1-p_{i.})}.$$

If we weight these by the quantities $(1-p_{i.})/(k-1)$ and sum, we obtain

$$\frac{1}{k-1}\sum_{i,j}\frac{(p_{ij}-p_{i.}p_{.j})^2}{p_{i.}p_{.j}} = \frac{\phi^2}{k-1}$$

in Karl Pearson's notation.

The mean-square contingency, like the second member of (14), depends on a knowledge of the p_{ij}; and while the sampling experiment will in due course provide estimates of these joint probabilities, they will not be available for an *a priori* assessment of the value of a proposed control variable. A study—not reproduced here—of the early work on the mean-square contingency (K. Pearson, 1913) suggests that if the quantitative variates x and y follow approximately a bivariate normal distribution with correlation ρ, and if the x and y scales are each divided into the same number $k = m$ of group intervals of equal probability, then the 'average variance reduction' should be given approximately by

$$\rho^2/\{k-(k-1)\rho^2\}. \tag{15}$$

Here, as above, the 'average variance reduction' is defined as the weighted average of the individual proportional variance reductions $\{\text{var}(p_{i.}) - \text{var}(\hat{p}_{i.})\}/\text{var}(p_{i.})$, with weighting coefficients $(1-p_{i.})/(k-1)$. Numerical examples that we have investigated indicate that formula (15) is sensitive to departures from the conditions stated above; nevertheless, it may on occasion serve as a useful guide at the start of a sampling experiment. It requires a knowledge only of the product-moment correlation between x and y, which if not known *a priori* may be estimated from a pilot experiment. As we would expect, (15) indicates that for $\rho^2 = 0$ there is no reduction in variance, while for $\rho^2 = 1$ the proportional reduction is unity, i.e. the variances of the $\hat{p}_{i.}$ are all zero.

7. It should be noted that the formula (15) indicates that for given ρ the proportional reduction in $\text{var}(p_{i.})$ is larger for small k, and for large k tends to zero. This is borne out, more generally, by numerical experiment on the basis of (14). Thus the method of using control variables will in general be more effective, if the number of frequency groups k is small, always provided that this reduction is otherwise acceptable. To illustrate this we may, for example, pool the frequencies in Table 1 to form, after division by the sample size, the two-by-two table:

	$0 \leqslant x \leqslant 2{\cdot}75$	$2{\cdot}75 \leqslant x \leqslant \infty$	Totals
$0 \leqslant y \leqslant 2$	0·3660	0·0205	0·3865
$2 \leqslant y \leqslant \infty$	0·0390	0·5745	0·6135
Totals	0·4050	0·5950	1·0000

The proportional reduction in variance now assumes the simple form

$$(p_{11}-p_{1.}p_{.1})^2/p_{1.}p_{.1}p_{2.}p_{.2},$$

which in the above example has the value 0·768, so that we have a relative efficiency of $100/(1-0{\cdot}768) = 431\,\%$.

APPENDIX

The first negative moment of a binomial variate

In the preceding paper (see equations (8) and (13)) we have quoted an approximation to the sum

$$E'(np, n) = \sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} p^r q^{n-r}, \tag{1}$$

which is the expectation of the reciprocal of a binomial variate, excluding the zero class. For small values of n, E' can easily be computed directly from existing tables of binomial terms (National Bureau of Standards, 1952).* In this Appendix we discuss its evaluation for large values of n, arriving in particular at the approximation already mentioned.

It is convenient to consider first the limiting case of the Poisson distribution, with mean m say, for which the analogous sum is

$$E'(m, \infty) = e^{-m} \sum_{r=1}^{\infty} \frac{1}{r}\frac{m^r}{r!} = e^{-m}\int_0^m \frac{e^t-1}{t}\,dt. \tag{2}$$

This integral may be expressed (Bromwich, 1926) as

$$\int_{-\infty}^{m} - \int_{-\infty}^{0} \frac{e^t-1}{t}\,dt = \mathrm{Ei}\,(m) - \log_e m - \gamma,$$

where γ is Euler's constant 0·57721 ..., and $\mathrm{Ei}\,(m)$ is the logarithmic integral

$$\mathrm{Ei}\,(m) = \int_{-\infty}^{m} \frac{e^t}{t}\,dt.$$

Thus $E'(m, \infty)$ may be calculated as

$$E'(m, \infty) = e^{-m}\{\mathrm{Ei}\,(m) - \log_e m - \gamma\} \tag{3}$$

from the tables of $\mathrm{Ei}\,(x)$ provided by the British Association (1931) or by Jahnke & Emde (1952).

Replacing the logarithmic integral in (3) by its well-known asymptotic expansion gives, for large m,

$$E'(m, \infty) + e^{-m}(\log_e m + \gamma) = \frac{1}{m} + \frac{1}{m^2} + \frac{2!}{m^3} + \ldots, \tag{4}$$

a series that yields maximum accuracy if we retain its terms until they start increasing.

Returning to the binomial distribution, we may clearly rewrite (1) as

$$E'(np, n) = q^n \int_0^{p/q} \frac{(1+u)^n - 1}{u}\,du. \tag{5}$$

The substitution $u = t/n$ gives $\quad E'(np, n) = q^n \int_0^{np/q} \frac{(1+t/n)^n - 1}{t}\,dt,$

a form the limit of which, if $n \to \infty$ and $p \to 0$ so that $np \to m$, is the integral in (2). The alternative substitution $u = (p-t)/q$ transforms (5) to

$$E'(np, n) = -n \int_0^p \frac{(1-t)^n - q^n}{p-t}\,dt,$$

or, on integrating by parts,

$$E'(np, n) = -n \int_0^p (1-t)^{n-1} \log(1-t/p)\,dt, \tag{6}$$

since the additional term $\{(1-t)^n - q^n\}\{\log(p-t) - \log p\}$ resulting from the partial integration vanishes at both limits.

* Since this note was first submitted, Grab & Savage (1954) have published tables covering the ranges $n = 2\,(1)\,30$ for the binomial and $m = 0{\cdot}01,\ 0{\cdot}05\ (\cdot05)\ 1{\cdot}00\ (\cdot10)\ 1{\cdot}80$ for the Poisson distribution.

Expanding the logarithmic term in (6), we express E' as a series of incomplete beta-functions*

$$E'(np, n) = \frac{1}{p(n+1)} I_p(2, n) + \frac{1!}{p^2(n+2)^{(2)}} I_p(3, n) + \frac{2!}{p^3(n+3)^{(3)}} I_p(4, n) + \ldots, \tag{7}$$

where $s^{(t)}$ stands, as usual, for $s!/(s-t)!$. For moderate values of n, $E'(np, n)$ might be calculated from (7) with the help of the available tables (Pearson, 1934); for large values of n and those values of r for which $r/n \ll p$, $I_p(r, n)$ is almost unity, so that if we write m for np, (7) suggests, in analogy with (4),

$$E'(m, n) \simeq \frac{1}{m+p} + \frac{1!}{(m+p)(m+2p)} + \frac{2!}{(m+p)(m+2p)(m+3p)} + \ldots. \tag{8}$$

This result is established more rigorously below; we remark now that it can easily be verified that as far as terms of order $(m-q)^{-4}$, (8) is equivalent to

$$E'(m, n) = \frac{1}{m-q} + \frac{q}{(m-q)^3} + \frac{q(q+1)}{(m-q)^4} = \frac{1}{m-q} + \frac{q(m+1)}{(m-q)^4}, \tag{9}$$

a convenient form when m is large and q small.

To obtain a more accurate asymptotic expression for E', we consider the sum of the first s terms in (7), which may be written as

$$E'_s(np, n) = np \int_0^1 (1-pt)^{n-1} \left(t + \tfrac{1}{2}t^2 + \ldots + \frac{t^s}{s}\right) dt. \tag{10}$$

A first integration by parts gives

$$E'_s(np, n) + q^n\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{s}\right) = \int_0^1 (1-pt)^n (1 + t + \ldots + t^{s-1})\, dt;$$

after u further integrations by parts we shall arrive at an integral term

$$\frac{1}{(n+u)^{(u)} p^u} \int_0^1 (1-pt)^{n+u} \{u! + (u+1)^{(u)} t + \ldots + (s-1)^{(u)} t^{s-u-1}\}\, dt,$$

and after one more integration at an additional term

$$\frac{-1}{(n+u+1)^{(u+1)} p^{u+1}} [(1-pt)^{n+u+1} \{u! + (u+1)^{(u)} t + \ldots + (s-1)^{(u)} t^{s-u-1}\}]_0^1$$

$$= \{u! - s^{(u+1)} q^{n+u+1}/(u+1)\}/p^{u+1}(n+u+1)^{(u+1)}.$$

It follows that

$$E'_s(np, n) + q^n\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{s}\right) = \sum_{u=1}^{s} [\{(u-1)! - s^{(u)} q^{n+u}/u\}/p^u (n+u)^{(u)}]. \tag{11}$$

If we write m for np, take $s = m_1$, the largest integer not exceeding m, replace the harmonic sum in (11) by its asymptotic value and ignore the difference between E'_{m_1} and E', we arrive at the analogue to (4)

$$E'(m, n) + q^n\{(\log_e m_1 + \gamma) + \sum_{u=1}^{m_1} (q/p)^u m_1^{(u)}/u(n+u)^{(u)}\}$$

$$\simeq \frac{1}{m+p} + \frac{1!}{(m+p)(m+2p)} + \frac{2!}{(m+p)(m+2p)(m+3p)} + \ldots \text{ to } m_1 \text{ terms}. \tag{12}$$

Since $$\sum_{u=1}^{m_1} (q/p)^u m_1^{(u)}/u(n+u)^{(u)} < \sum_{u=1}^{\infty} q^u/u = -\log_e p,$$

the second term in the first member of (12) is certainly less than $q^n(\log_e n + \gamma)$.

The entries in the accompanying table of E', computed by Miss Pamela Johnson, were obtained as follows:

$m = 1(1)10$,	$n = 25, 50$	directly from equation (1);
$m = 12(2)20$,	$n = 25, 50, 100$	from equation (8),
$25(5)45$,	$50, 100$	controlled by equation (12);
$m = 1(1)10(2)14$,	$n = \infty$	from equation (3),
$m = 16(2)20(5)45$,	$n = \infty$	from equation (4).

* Stephan (1945) has given a result equivalent to (7), with the incomplete beta-functions replaced by the corresponding binomial tails; he has also shown that the series in (7) converges, and how it may be used for accurate calculation.

For the values of m listed, and intermediate values of n, E' may be found by harmonic interpolation. For $m \geqslant 1$ and $n \geqslant 100$, $E'(m, \infty)$, calculated from equation (3) or (4), clearly provides a reasonably good working approximation to $E'(m, n)$. For $m > 4$, this approximation may be improved by linear interpolation, with $p = m/n$ as interval, between $E'(m, \infty)$ and $E'(m, m) = 1/m$; the error in the interpolate for $m = 5$, $n = 100$, for example, is about 0·0004, and for $m = 25$, $n = 100$ about 0·00002.

Values of $E'(np, n) = \sum_{r=1}^{n} \frac{1}{r} \binom{n}{r} p^r q^{n-r}$

m	$n=25$	$n=50$	$n=\infty$	m	$n=25$	$n=50$	$n=100$	$n=\infty$
1	0·4932	0·4890	0·4848	12	0·08752	0·08964	0·09074	0·09190
2	·5051	·5018	·4986	14	·07395	·07567	·07656	·07749
3	·4096	·4105	·4111	16	·06404	·06549	·06624	·06702
4	·3160	·3200	·3236	18	·05649	·05774	·05839	·05906
5	·2460	·2511	·2560	20	·05053	·05164	·05221	·05280
6	0·1972	0·2023	0·2073	25	0·04000	0·04085	0·04129	0·04175
7	·1632	·1678	·1723	30	—	·03380	·03416	·03453
8	·1390	·1429	·1468	35	—	·02883	·02913	·02944
9	·1211	·1243	·1277	40	—	·02513	·02539	·02566
10	·1073	·1101	·1130	45	—	·02227	·02250	·02274

For large $m = np$, $E' \simeq \frac{1}{m-q} + \frac{q}{(m-q)^3} + \frac{q(q+1)}{(m-q)^4}$.

This paper was in part prepared in the Statistical Advisory Unit, Ministry of Supply, and is published with the permission of the Ministry's Chief Scientist.

REFERENCES

BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE (1931). *Mathematical Tables*, **1**, Table VII.

BROMWICH, T. J. I. A. (1926). *An Introduction to the Theory of Infinite Series*, 2nd ed. London: Macmillan and Co.

GRAB, E. L. & SAVAGE, I. R. (1954). *J. Amer. Statist. Ass.* **49**, 169.

JAHNKE, E. & EMDE, F. (1952). *Tables of Higher Functions*, 5th ed. Leipzig: Teubner.

NATIONAL BUREAU OF STANDARDS (1952). *Tables of the Binomial Probability Distribution*. Washington: U.S. Government Printing Office.

NEYMAN, J. (1938). *J. Amer. Statist. Ass.* **33**, 101.

PEARSON, E. S. & HARTLEY, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge University Press for the Biometrika Trustees.

PEARSON, K. (1913). *Biometrika*, **9**, 116.

PEARSON, K. (1934). *Tables of the Incomplete Beta Function*. Cambridge University Press for the Biometrika Trustees.

STEPHAN, F. F. (1945). *Ann. Math. Statist.* **16**, 50.

WOLD, H. (1948). *Random Normal Deviates*. Tracts for Computers, no. XXV. Cambridge University Press.

RANK ANALYSIS OF INCOMPLETE BLOCK DESIGNS

II. ADDITIONAL TABLES FOR THE METHOD OF PAIRED COMPARISONS

By RALPH ALLAN BRADLEY*

Virginia Agricultural Experiment Station of the Virginia Polytechnic Institute

1. Introduction and summary

The purpose of this paper is to provide additional tables for the method of paired comparisons introduced by Bradley & Terry (1952), and it is not our intention to review the theoretical aspects of that method. An illustration of the use of the method is included in the cited reference, and a more detailed application was presented by Terry, Bradley & Davis (1952) for the use of food technologists. Additional research has been conducted by Bradley (1954) on the appropriateness of the model for paired comparisons and by Abelson & Bradley (1954) on factorial arrangements of treatments in paired comparisons.

In paired comparisons we use t to denote the number of treatments or items to be compared, and we define a repetition to be a single set of the $\binom{t}{2}$ treatment comparisons. A repetition may then be regarded to be a single set of incomplete blocks of size two with pairs of treatments appearing together once and only once in the incomplete blocks of the set. The number of repetitions in a paired comparisons experiment is represented by n. Tables for paired comparisons for $t = 3$, $n = 1, \ldots, 10$, and for $t = 4$, $n = 1, \ldots, 6$ were included in Appendix A of the initial paper on the subject (Bradley & Terry, 1952). For experiments in which ranks are assigned to the treatments within the incomplete blocks, these tables show the distribution of a statistic, B_1, used for a test of treatment equality, along with estimates, $p_1, \ldots, p_t$, of postulated treatment parameters, or ratings, $\pi_1, \ldots, \pi_t$. When a single set of treatment parameters is sufficient to characterize the rankings in all repetitions of the experiment, we say that the n repetitions are a homogeneous set. If the treatment parameters are permitted to differ for different repetitions or groups of repetitions of the experiment, one is in effect admitting the possibility of 'treatment by repetition' or 'treatment by groups of repetitions' interaction. The tables of Appendix A provide means for a valid test of treatment equality under the assumption that the n repetitions in the experiment are a homogeneous set. A second table, that in Appendix B of the same paper, provides some significance levels for B_1^c, the combined likelihood-ratio statistic, which was formulated to provide a test procedure for a test of treatment equality, admitting possible treatment by repetitions (or groups of repetitions) interaction. The detection of treatment by repetition interaction may be accomplished using the test of agreement previously presented.

Interest in and extensive use of the method of paired comparisons formulated has encouraged the author to provide additional tables. In this paper the table of Appendix A has been extended for $t = 4$ to $n = 7, 8$, and to $t = 5$, $n = 1, \ldots, 5$. These tables are volu-

* This paper reports research undertaken in co-operation with the Quartermaster Food and Container Institute for the Armed Forces and has been assigned number 454 in the series of papers approved for publication. The views or conclusions contained in this report are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Department of Defence of the United States of America.

minous, and the cost of further extensions was prohibitive with the computing equipment and resources available. However, these tables may be sufficient for, with $t = 3, 4$, or 5, it appears that the approximate methods previously presented will be adequate for values of n beyond the range of the tables, and, for more than five treatments, it may be more desirable to use incomplete blocks of size greater than two. No additional tables for B_1^* could be computed, and again approximate methods may often be used.

In addition to the new tables in the Appendix of this paper, we include some notes on the uses of the tables and some minor corrections to the previously published tables. The statistic B_1 is a monotone function of a likelihood-ratio statistic which has been shown to have a limiting χ^2-distribution. For all of the available tables we have computed the actual means and variances of the corresponding likelihood-ratio statistics and the convergence of these values to their limits is illustrated in Table 1. Certain checks on the computing were possible, and these have been noted as of interest to the reader.

2. USES OF THE TABLES

Parameters, $\pi_1, \ldots, \pi_t$, with $\pi_i \geqslant 0$, $i = 1, \ldots, t$, $\sum_{i=1}^{t} \pi_i = 1$, are associated with t treatments in paired comparisons. These parameters are further defined with the assumption that, when treatment i appears with treatment j in a comparison, the probability that treatment i obtains top rating (or a rank of 1) is $\pi_i/(\pi_i + \pi_j)$. Maximum-likelihood estimators of these parameters are respectively $p_1, \ldots, p_t$. Analysis of paired comparisons depends on $\Sigma r_1, \ldots, \Sigma r_t$, the set of total sums of ranks for the t treatments after ranking has been effected independently within the blocks of size two (the rank 1 was assigned to the 'better' of the two treatments in a pair and the rank 2 to the 'poorer'). The statistic B_1, for tests of treatment equality assuming homogeneous repetitions of paired comparisons, has the form*

$$B_1 = n \sum_{i<j} \log (p_i + p_j) - \sum_i \{2n(t-1) - \Sigma r_i\} \log p_i. \tag{1}$$

The tables in the Appendix show all possible sets of sums of ranks, $\Sigma r_1, \ldots, \Sigma r_t$, for specified values of t and n, the corresponding sets of estimates, $p_1, \ldots, p_t$, values of B_1, and the significance levels of B_1. It has been indicated (Bradley, 1953) that $\log p_1, \ldots, \log p_t$ may be regarded as estimators of treatment locations on an additive continuum.

The basic purpose of the prepared tables is the analysis of paired comparison experiments with homogeneous repetitions. However, for this purpose alone and with only the objective of determining significance levels for the test of treatment equality, it would be sufficient to list the sets of sums of ranks and their significance levels. To extend consideration to the problem of estimation, we also list the estimates of treatment parameters, $p_1, \ldots, p_t$. It is useful to list these estimates since their computation is troublesome. The statistic B_1 is used to order the rows in the tables and is not otherwise required in either the test or estimation. But values of B_1 have been given since they are used in certain approximate tests.

The statistic B_1 is required for:

(1) The combined test of treatment equality.
(2) The test for homogeneous repetitions.
(3) The test of the appropriateness of the model.
(4) The analysis of paired comparisons with treatments in factorial arrangements.

* We use log to represent a common logarithm and ln to represent a natural logarithm.

This list indicates some of the further uses of the tables which we shall discuss briefly.

(1) Suppose the experiment with n repetitions is subdivided into g groups of repetitions with n_u repetitions in the uth group, $u = 1, \ldots, g$. If the n repetitions were assumed to be homogeneous, treatment parameters could be limited to $\pi_1, \ldots, \pi_t$ and the grouping could be ignored. When group differences enter, we postulate parameters $\pi_{1u}, \ldots, \pi_{tu}$ for the uth group and allow the parameters to vary from group to group. To test for treatment equality with the more general alternative specification of the parameters, we use the statistic

$$B_1^c = \sum_{u=1}^{g} B_1^u, \tag{2}$$

where B_1^u is the statistic B_1 for the uth group. The tables are then used to obtain values of B_1^u for the computation of B_1^c. Limited tables for B_1^c have been published, and an approximate test depending on χ^2 with $g(t-1)$ degrees of freedom is available. Reference is to Bradley & Terry (1952, p. 332).

(2) Sometimes it is necessary to decide between the postulation of one set of parameters, $\pi_1, \ldots, \pi_t$, under the assumption of homogeneous repetitions and the use of g sets of parameters needed when repetitions are grouped. Decision is possible using a test of agreement which depends on

$$-2\ln\lambda = 2(B_1 - B_1^c)\ln 10. \tag{3}$$

This statistic has a distribution approximated by that of χ^2 with $(g-1)(t-1)$ degrees of freedom. B_1 is read from the tables for the total n repetitions and B_1^c also depends on the tables through (2). Reference is again to Bradley & Terry (1952, p. 332).

(3) Consider n homogeneous repetitions of paired comparisons and let us define f_{ij} to be the number of times treatment i obtains the rank 1 in direct comparison with treatment j. $f_{ij} + f_{ji} = n$. A test of the appropriateness of the model, or of goodness of fit, for paired comparisons has been devised and depends on the statistic

$$-2\ln\lambda_1 = 2\left[B_1 - n\binom{t}{2}\log n + \sum_{i<j} f_{ij}\log f_{ij}\right]\ln 10. \tag{4}$$

Again B_1 is required from the tables. Reference is to Bradley (1954).

(4) The analysis of paired comparison experiments with treatments in a factorial arrangement has been developed. At the present time easy analysis seems to be limited to the 2×2 factorial. Treatment parameters, $\pi_1, \ldots, \pi_4$, are replaced by the products of parameters, $\alpha_1\beta_1$, $\alpha_1\beta_2$, $\alpha_2\beta_1$ and $\alpha_2\beta_2$, $\alpha_1 + \alpha_2 = 1$, $\beta_1 + \beta_2 = 1$, $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, $i = 1, 2$, for treatments composed of A- and B-factors, each at one of two levels. The first step in the analysis depends on the statistic B_1 for an over-all test of treatment effects. The statistic used has a distribution approximated by that of χ^2 with three degrees of freedom. These three degrees of freedom may be partitioned in two ways, but in both cases one degree of freedom is used to measure A by B interaction. The measure of interaction is a statistic that depends again on B_1 along with maximum-likelihood estimators of α_1, α_2, β_1 and β_2. Reference is to Abelson & Bradley (1954).

In some cases n may exceed values for which tables have been prepared while the value of t is among those tabled. The available tables will still be of considerable assistance. In the special case where $\Sigma r_1, \ldots, \Sigma r_t$ and n are all multiples of an integer c, it is sufficient to enter the table for t treatments and n/c repetitions with sums of ranks, $\Sigma r_1/c, \ldots, \Sigma r_t/c$. Values of $p_1, \ldots, p_t$ may be read directly from the table, and the desired value of B_1 may be

obtained by multiplying the tabular value of B_1 by c. The test is completed by substituting the value of B_1 so determined in

$$-2\ln\lambda_1 = nt(t-1)\ln 2 - 2B_1\ln 10. \qquad (5)$$

This statistic has the distribution of χ^2 with $(t-1)$ degrees of freedom for large values of n. When n/c is an integer, and if tables are available for n/c repetitions and t treatments, even though $\Sigma r_i/c \ldots, \Sigma r_t/c$ are not all integers, the prepared tables may be used to interpolate for values of $p_1, \ldots, p_t$. The best results will be obtained when c is the smallest integer for which n/c is tabled. These values of $p_1, \ldots, p_t$ may be regarded as a first approximation, and the final values are computed using the iterative procedure outlined by Bradley & Terry (1952). It is then necessary to compute B_1 using (1) and to complete the approximate test using (5). Alternatively, one may interpolate for the value of B_1 with n/c repetitions, compute the required B_1 by again multiplying the tabular value by c, and substitute in (5) for the approximate test. This second alternative will be satisfactory if values of $p_1, \ldots, p_t$ are not required.

Apart from the direct applications of the tables to methods of paired comparisons outlined above, the tables give the distribution of possible sets of sums of ranks in paired comparisons. Under the hypothesis of treatment equality, all permutations of sums of ranks in a set are equally likely and their individual probabilities may be recovered from the cumulative probabilities tabled (sets of sums of ranks with identical values of B_1 also have identical probabilities). These tables may then be useful if new methods of paired comparison are devised and even though the new methods may re-order the sets of sums of ranks.

3. TABLE ACCURACY AND COMPUTING CHECKS

The method of computing employed has been outlined by Bradley & Terry (1952). Probabilities were obtained by the method indicated in Table 1 and estimates, $p_1, \ldots, p_t$, by the iterative procedure set forth in § 10 of that reference. All computations were done with the use of punch-cards and I.B.M. equipment with the exception of those on values of B_1 which were obtained using desk calculators. Values of $p_1, \ldots, p_t$ were obtained correct to four decimal places, but they have been rounded to two places to facilitate publication. Similarly, values of B_1 have been rounded to three places from initial computations which showed four decimal places. Probabilities or significance levels are correct to four decimal places as shown.

Certain checks on the computations are possible. In the tables certain entries are grouped. These sets of sums of ranks yield identical values of B_1 and also have identical probabilities of occurrence. Except for a few symmetric sets of sums of ranks, which stand as single-table entries, sets of sums of ranks may be paired and their values of B_1 should match. The relationship that exists between paired sets of sums of ranks is as follows: with n repetitions with t treatments, the set of sums of ranks, $\Sigma r_1, \ldots, \Sigma r_t$, has, as a mate, the set $3n(t-1)-\Sigma r_t$, $\ldots$, $3n(t-1)-\Sigma r_1$. For all such paired sets of sum of ranks, values of $p_1, \ldots, p_t$ and B_1 were computed separately for each member of the pair and the resultant values of B_1 were compared. Some sets of sums of ranks are symmetric and have no mates. This occurs when

$$3n(t-1) - \Sigma r_{t-i+1} = \Sigma r_i \quad (i = 1, \ldots, t).$$

For such table entries, all computations were independently repeated.

As a secondary check on the tabling, the sum of squares of the sums of ranks in each set was computed. This statistic does not order the sets of sums of ranks exactly as does B_1, but the orderings agree quite closely. Thus, after ordering using B_1, if we found a value of a sum of squares much out of order, the calculation leading to the corresponding value of B_1 was checked. All sets of sums of ranks grouped together in the tables have equal values of the sums of squares of sums of ranks.

The number of possible decisions on rankings in a paired comparison experiment is $2^{\frac{1}{2}nt(t-1)}$, and the number of these leading to an ordered set of sums of ranks was used to compute the probability of obtaining such a set under the hypothesis of treatment equality. These frequencies were used to obtain the means and variances of B_1 and thence of $-2\ln\lambda_1$ given in (5). The results, based then on the tables, are given in Table 1 of this paper and indicate the convergence of these means and variances to the corresponding means and variances of the appropriate χ^2-approximation. This table has two purposes. It gives some assurance in the use of the approximate test for values of n exceeding those in the tables. It also has been of some use as a further check on the computations and this led to one or two corrections to the tables.

Table 1. *Means and variances of* $-2\log_e\lambda_1$

n	$t=3$		$t=4$		$t=5$	
	Mean	Variance	Mean	Variance	Mean	Variance
1	3·12	3·24	4·55	9·96	5·61	16·06
2	3·39	7·27	3·59	9·51	4·58	11·13
3	2·80	7·50	3·33	7·80	4·32	9·57
4	2·54	6·51	3·22	7·10	4·23	9·01
5	2·40	5·83	3·17	6·76	4·18	8·77
6	2·32	5·38	3·14	6·61	—	—
7	2·27	5·15	3·12	6·51	—	—
8	2·23	4·95	3·11	6·45	—	—
9	2·20	4·82	—	—	—	—
10	2·18	4·71	—	—	—	—
∞	2·00	4·00	3·00	6·00	4·00	8·00

4. Extreme sets of sums of ranks

The smallest sum of ranks that a treatment may have with t treatments and n repetitions is $n(t-1)$. When this value occurs, the set of estimates, $p_1, \ldots, p_t$, has values, $1, 0, \ldots, 0$. Substitution of these estimates in (1) for B_1 yields an indeterminate form. B_1 may, however, be computed by assuming that the estimates are $1, a_1\delta, \ldots, a_{t-1}\delta$. Then it is possible to determine B_1 as a limit as $\delta \to 0$. Values of $a_1, \ldots, a_{t-1}$ are obtained by maximizing the appropriate likelihood function.

It turns out that the normal equations for $a_1, \ldots, a_{t-1}$ in the maximization process have already been solved in the computation for $(t-1)$ treatments and n repetitions. Further, the value of B_1 also occurs in that table. The procedure in this case is simply to obtain the reduced set of $(t-1)$ sums of ranks, $\Sigma r_2 - 2n, \ldots, \Sigma r_t - 2n$, and to enter the table specified with this set of sums of ranks. The required value of B_1 is copied from that entry, and we

could also record the treatment parameter estimates as secondary estimates of the a's for our new table.

These notes can be illustrated. With $t = 5, n = 5$, consider the set of sums of ranks 20, 29, 30, 32, 39. The table shows estimates $1, 0, \ldots, 0$ and $B_1 = 5{\cdot}858$. The reduced set of sums of ranks is 19, 20, 22, 29. The table (in the 1952 paper) for $t = 4, n = 5$ contains this entry with estimates, 0·44, 0·34, 0·20, 0·02 and $B_1 = 5{\cdot}858$. In the table for $t = 5, n = 5$, we could have shown the estimates as 1, (0·44), (0·34), (0·20), (0·02), the secondary estimates being in parentheses. The secondary estimates are of course equivalent to those that would be found if the data were analysed ignoring all comparisons involving the first treatment with the extreme sum of ranks. This only means that this first treatment is so 'superior' to the others that differences among the remaining treatments are overshadowed by its presence.

Another extreme case of less consequence occurs when one of t treatments obtains the maximum possible sum of ranks, $2n(t-1)$. Then, again, a reduced set of sums of ranks may be obtained for the remaining $(t-1)$ treatments. This reduced set is $\Sigma r_1 - n, \ldots, \Sigma r_{t-1} - n$, and the table for $(t-1)$ treatments and n repetitions may be used to find $p_1, \ldots, p_{t-1}$ and B_1 directly. In this case p_t has the value zero.

These two kinds of extreme sets of sums of ranks indicate yet another use of the tables. In the special cases where extreme sets of sums of ranks occur, and where t exceeds tabled values by unity, estimates $p_1, \ldots, p_t$ and B_1 may be obtained with the help of the table for $(t-1)$ treatments and n repetitions. The approximate test may then be used by computing $-2\ln\lambda_1$ given in (5).

5. Errata in previous tables

Only two errors have been noted in the tables published previously and these are both trivial.

(1) In the table for $t = 4, n = 5$, the value of B_1 recorded as 1·474 should be 2·173. The table entry then reads:

16, 19, 26, 29; ·80, ·20, —, —; 2·173; —.

This correction does not change the order of the recorded sets of sums of ranks.

(2) In the table for $t = 4, n = 5$, the rows with $B_1 = 7{\cdot}884$ and 7·895 should be interchanged. The probabilities P must then be adjusted and these two rows should read:

19,	22,	23,	26;	·47,	·24,	·19,	·10;	7·884;	·1909
20,	20,	25,	25;	·37,	·37,	·12,	·12;	7·895;	·1964.

The values for the mean and variance of $-2\ln\lambda_1$ given in Table 2 of the earlier paper for $t = 4, n = 5$ have been corrected and are recorded in Table 1 of this paper. The mean is corrected from 3·13 to 3·17 and the variance from 6·66 to 6·76.

Funds for this computing project were provided by the Quartermaster Food and Container Institute for the Armed Forces. In addition to this assistance, the author would like to acknowledge the co-operation of Dr D. K. Tressler, Scientific Director of that Institute, and of K. R. Wood, D. R. Peryam and Miss Dorothy Johnson. The original paper on paired comparisons was prepared under Contract No. A-1s-32683 with the Bureau of Agricultural Economics. United States Department of Agriculture, and the continued interest of members of that Department and of M. E. Terry, who was co-author of that paper, is appreciated.

The computation of the tables could not have been completed without the direct assistance of quite a large number of people. The author is indebted to R. G. Cornell for supervising much of the computing, to R. M. Abelson, who primarily assisted Mr Cornell, to A. F. Teske and Mrs Elizabeth Skinner, who made the facilities of the I.B.M. Tabulating Division available and provided assistance and helpful advice, and to C. Y. Kramer, R. E. Kirk, Miss Nancy Phillippi, Mrs Marianne Byrd, Mrs Ganelle Swartz, Mrs Willie S. Dickerson and Mrs Joan Ledwith for hand computations and clerical assistance.

REFERENCES

Abelson, R. M. & Bradley, R. A. (1954). A 2×2 factorial with paired comparisons. (To be published in *Biometrics*.)

Bradley, R. A. (1953). Some statistical methods in taste testing and quality evaluation. *Biometrics*, **9**, 22.

Bradley, R. A. (1954). Incomplete block rank analysis: on the appropriateness of the model for a method of paired comparisons. (To be published in *Biometrics*.)

Bradley, R. A. & Terry, M. E. (1952). Rank analysis of incomplete block designs. I. The method of paired comparisons. *Biometrika*, **39**, 324.

Cochran, W. G. & Cox, Gertrude (1950). *Experimental Designs*. New York: John Wiley and Sons, Inc.

Fisher, R. A. & Yates, F. (1953). *Statistical Tables for Biological, Agricultural and Medical Research*, 4th ed. Edinburgh: Oliver and Boyd.

Terry, M. E., Bradley, R. A. & Davis, L. L. (1952). New designs and techniques for organoleptic testing. *Food Technology*, **6**, 250.

TABLES FOR THE RANK ANALYSIS OF INCOMPLETE BLOCK DESIGNS

Appendix. *The distribution of the likelihood ratio for general alternatives*

The following table gives the values of the likelihood-ratio statistic, B_1, and the likelihood estimates of the true treatment ratings, $p_1, \ldots, p_t$, together with probabilities, P, that B_1 will not be exceeded if the null hypothesis is true. Since low values of B_1 indicate discordant results, P gives the significance level.

n is the number of repetitions of the design and t the number of treatments. The design symbols, t or v, λ, b, r, k are standard and as used, for example, by Fisher & Yates (1953, p. 20, Introduction to Tables XVII–XVIII) and Cochran & Cox (1950, pp. 270 and 304). λ in this design description should not be confused with the same symbol generally used to indicate a likelihood ratio. Parentheses contain combinations with equal values of B_1. Σr_i is the sum of ranks for treatment i.

In setting up the table, several conventions have been adopted to simplify the printing: (i) When p_i is unity and the remaining estimators are therefore all zero, the result is given as 1 — — — or 1 —‹— — —. (ii) The lowest value of B_1 possible for each n is zero and is as 0. (iii) When there are no entries in the final column above a single entry of ·0000, the corresponding values of P are less than a half-unit in the fifth decimal place.

4 treatments. (*Design*: $t = 4, \lambda = 1, b = 6, r = 3, k = 2$)

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n=7$					
21	28	35	42	1	—	—	—	0	
21	28	36	41	1	—	—	—	1·247	
21	29	34	42	1	—	—	—		
22	27	35	42	·86	·14	—	—		
21	28	37	40	1	—	—	—	1·819	
21	30	33	42	1	—	—	—		
23	26	35	42	·71	·29	—	—		
21	28	38	39	1	—	—	—	2·076	
21	31	32	42	1	—	—	—		
24	25	35	42	·57	·43	—	—		
22	27	36	41	·86	·14	—	—	2·494	
21	29	35	41	1	—	—	—	2·563	
22	28	34	42	·86	·12	·02	—		
22	27	37	40	·86	·14	—	—	3·066	
23	26	36	41	·71	·29	—	—		
21	29	36	40	1	—	—	—	3·213	
21	30	34	41	1	—	—	—		
22	29	33	42	·86	·10	·03	—		
23	27	34	42	·73	·24	·03	—		
22	27	38	39	·86	·14	—	—	3·323	
24	25	36	41	·57	·43	—	—		
21	29	37	39	1	—	—	—	3·561	
21	31	33	41	1	—	—	—		
22	30	32	42	·87	·08	·05	—		
24	26	34	42	·60	·36	·03	—		
23	26	37	40	·71	·29	—	—	3·638	
21	29	38	38	1	—	—	—	3·672	
21	32	32	41	1	—	—	—		
22	31	31	42	·87	·07	·07	—		
25	25	34	42	·48	·48	·04	—		
22	28	35	41	·86	·12	·01	—	3·878	
23	26	38	39	·71	·29	—	—	3·895	
24	25	37	40	·57	·43	—	—		
21	30	35	40	1	—	—	—	3·942	
23	28	33	42	·74	·20	·06	—		
24	25	38	39	·57	·43	—	—	4·152	
21	30	36	39	1	—	—	—	4·372	
21	31	34	40	1	—	—	—		
23	29	32	42	·75	·17	·08	—		
24	27	33	42	·62	·31	·07	—		
22	28	36	40	·86	·12	·01	—	4·527	
23	27	35	41	·73	·24	·02	—		
21	30	37	38	1	—	—	—	4·576	
21	32	33	40	1	—	—	—		
23	30	31	42	·75	·14	·11	—		
25	26	33	42	·51	·41	·08	—		
22	29	34	41	·86	·10	·03	—	4·600	
22	28	37	39	·86	·12	·01	·01	4·874	
24	26	35	41	·60	·36	·03	—		
21	31	35	39	1	—	—	—	4·884	
24	28	32	42	·64	·26	·10	—		
22	28	38	38	·86	·12	·01	0·1	4·984	
25	25	35	41	·48	·48	·03	—		
22	30	33	41	·87	·09	·04	—	5·027	
21	31	36	38	1	—	—	—	5·173	
21	32	34	39	1	—	—	—		
24	29	31	42	·64	·22	·14	—		
25	27	32	42	·54	·35	·12	—		
23	27	36	40	·73	·24	·02	·01	5·175	
22	31	32	41	·87	·07	·06	—	5·229	
21	31	37	37	1	—	—	—	5·266	
21	33	33	39	1	—	—	—		
24	30	30	42	·65	·18	·18	—		
26	26	32	42	·44	·44	·12	—		
22	29	35	40	·86	·10	·02	·01	5·323	
23	28	34	41	·74	·21	·05	·01		
23	27	37	39	·73	·24	·02	·01	5·521	
24	26	36	40	·60	·36	·03	·01		
21	32	35	38	1	—	—	—	5·544	
25	28	31	42	·55	·29	·16	—		
23	27	38	38	·73	·24	·01	·01	5·631	
25	25	36	40	·48	·48	·03	·01		
21	32	36	37	1	—	—	—	5·723	
21	33	34	38	1	—	—	—		
25	29	30	42	·55	·25	·20	—		
26	27	31	42	·46	·37	·17	—		
22	29	36	39	·87	·10	·02	·01	5·751	
24	27	34	41	·63	·31	·06	·01		
22	30	34	40	·87	·09	·04	·01	5·826	
23	29	33	41	·75	·17	·07	·01		
24	26	37	39	·61	·36	·02	·01	5·868	

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
					$n=7$ (*cont.*)				
22	29	37	38	·87	·10	·02	·01	5·953	
25	26	34	41	·52	·41	·07	·01		
24	26	38	38	·61	·36	·02	·02	5·978	
25	25	37	39	·48	·48	·02	·01		
21	33	35	37	1	—	—	—	5·984	
26	28	30	42	·47	·32	·21	—		
23	28	35	40	·74	·21	·04	·01	6·043	
21	33	36	36	1	—	—	—	6·070	
21	34	34	37	1	—	—	—		
26	29	29	42	·47	·26	·26	—		
27	27	30	42	·39	·39	·22	—		
25	25	38	38	·48	·48	·02	·02	6·088	
22	31	33	40	·87	·07	·05	·01	6·110	
23	30	32	41	·75	·15	·10	·01		
22	32	32	40	·87	·06	·06	·01	6·202	
23	31	31	41	·75	·12	·12	·01		
21	34	35	36	1	—	—	—	6·238	
27	28	29	42	·40	·33	·27	—		
21	35	35	35	1	—	—	—	6·322	
28	28	28	42	·33	·33	·33	—		
22	30	35	39	·87	·09	·03	·01	6·331	
24	28	33	41	·64	·26	·09	·01		
23	28	36	39	·74	·21	·03	·02	6·469	
24	27	35	40	·63	·31	·05	·01		
22	30	36	38	·87	·09	·03	·02	6·616	
25	27	33	41	·54	·35	·10	·01		
23	29	34	40	·75	·18	·06	·01	6·618	
23	28	37	38	·74	·21	·03	·02	6·670	
25	26	35	40	·52	·41	·06	·02		
22	31	34	39	·87	·08	·04	·01	6·693	
24	29	32	41	·65	·22	·12	·01		
22	30	37	37	·87	·09	·02	·02	6·708	
26	26	33	41	·44	·44	·10	·01		
22	32	33	39	·87	·06	·05	·02	6·867	
24	30	31	41	·65	·19	·15	·01		
24	27	36	39	·63	·31	·04	·02	6·893	
23	30	33	40	·75	·15	·08	·02	6·976	
22	31	35	38	·87	·08	·04	·02	7·055	
25	28	32	41	·55	·30	·14	·01		
24	27	37	38	·63	·31	·04	·03	7·094	
25	26	36	39	·52	·41	·05	·02		
23	29	35	39	·75	·18	·05	·01	7·118	
24	28	34	40	·64	·26	·08	·02		
23	31	32	40	·75	·13	·10	·02	7·149	

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
					$n=7$ (*cont.*)				
22	31	36	37	·87	·08	·03	·02	7·230	
26	27	32	41	·46	·38	·14	·01		
25	26	37	38	·52	·41	·04	·03	7·294	
22	32	34	38	·87	·07	·04	·02	7·308	
25	29	31	41	·56	·25	·17	·01		
22	33	33	38	·87	·06	·06	·02	7·350	
25	30	30	41	·56	·21	·21	·01		
23	29	36	38	·75	·18	·04	·03	7·400	
25	27	34	40	·54	·35	·09	·02		
23	29	37	37	·75	·18	·04	·04	7·491	0·0000
26	26	34	40	·44	·44	·09	·02		
23	30	34	39	·75	·15	·07	·02	7·550	·0001
24	29	33	40	·65	·23	·10	·02		
22	32	35	37	·88	·07	·04	·03	7·561	·0001
26	28	31	41	·48	·32	·19	·01		
24	28	35	39	·64	·27	·07	·03	7·615	·0001
22	32	36	36	·87	·07	·03	·03	7·644	·0001
27	27	31	41	·40	·40	·19	·01		
22	33	34	37	·87	·06	·05	·03	7·722	·0001
26	29	30	41	·48	·27	·23	·01		
23	31	33	39	·76	·13	·09	·03	7·798	·0001
24	30	32	40	·65	·19	·13	·02		
23	32	32	39	·76	·11	·11	·03	7·879	·0001
24	31	31	40	·66	·16	·16	·02		
22	33	35	36	·87	·06	·04	·03	7·884	·0001
27	28	30	41	·41	·34	·24	·02		
24	28	36	38	·64	·27	·06	·04	7·895	·0001
25	27	35	39	·54	·35	·07	·03		
23	30	35	38	·75	·15	·06	·03	7·906	·0001
25	28	33	40	·55	·30	·12	·02		
22	34	34	36	·87	·05	·05	·03	7·963	·0002
27	29	29	41	·41	·29	·29	·02		
24	28	37	37	·64	·27	·05	·05	7·986	·0002
26	26	35	39	·45	·45	·08	·03		
22	34	35	35	·87	·05	·04	·04	8·042	·0002
28	28	29	41	·35	·35	·29	·02		
23	30	36	37	·75	·15	·05	·04	8·079	·0002
26	27	33	40	·47	·38	·12	·03		
24	29	34	39	·65	·23	·09	·03	8·117	·0002
25	27	36	38	·54	·36	·06	·04	8·174	·0002
23	31	34	38	·76	·13	·08	·04	8·229	·0002
25	29	32	40	·56	·26	·15	·03		
25	27	37	37	·54	·36	·05	·05	8·265	·0003
26	26	36	38	·45	·45	·06	·04		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n=7$ (*cont.*)					
26	26	37	37	·45	·45	·05	·05	8·355	·0003
23	32	33	38	·76	·11	·09	·04	8·386	·0003
25	30	31	40	·57	·22	·18	·03		
24	30	33	39	·66	·20	·11	·03	8·436	·0003
24	29	35	38	·65	·23	·08	·04	8·470	·0004
25	28	34	39	·56	·31	·10	·04		
23	31	35	37	·76	·13	·07	·04	8·477	·0004
26	28	32	40	·48	·33	·16	·03		
23	31	36	36	·76	·13	·05	·05	8·558	·0005
27	27	32	40	·40	·40	·16	·03		
24	31	32	39	·66	·17	·14	·04	8·591	·0005
24	29	36	37	·65	·23	·06	·05	8·641	·0005
26	27	34	39	·47	·39	·11	·04		
23	32	34	37	·76	·11	·08	·05	8·708	·0006
26	29	31	40	·49	·28	·20	·03		
23	33	33	37	·76	·10	·10	·05	8·784	·0007
26	30	30	40	·49	·24	·24	·03		
25	28	35	38	·56	·31	·09	·05	8·815	·0007
24	30	34	38	·66	·20	·10	·05	8·860	·0008
25	29	33	39	·57	·26	·13	·04		
23	32	35	36	·76	·12	·07	·06	8·865	·0009
27	28	31	40	·42	·35	·21	·03		
25	28	36	37	·56	·31	·07	·06	8·991	·0011
26	27	35	38	·47	·39	·09	·05		
23	33	34	36	·76	·10	·08	·06	9·015	·0012
27	29	30	40	·42	·30	·25	·03		
24	31	33	38	·66	·17	·12	·05	9·087	·0014
25	30	32	39	·57	·23	·16	·04		
23	33	35	35	·76	·10	·07	·07	9·091	·0015
28	28	30	40	·36	·36	·25	·03		
24	30	35	37	·66	·20	·08	·06	9·105	·0017
26	28	33	39	·48	·33	·14	·04		
26	27	36	37	·47	·39	·08	·06	9·160	·0017
24	32	32	38	·66	·14	·14	·05	9·162	·0018
25	31	31	39	·57	·19	·19	·04		
23	34	34	35	·76	·08	·08	·07	9·165	·0019
28	29	29	40	·36	·30	·30	·03		
24	30	36	36	·66	·20	·07	·07	9·185	·0021
27	27	33	39	·41	·41	·14	·04		
25	29	34	38	·57	·27	·11	·05	9·280	·0022
24	31	34	37	·66	·17	·10	·06	9·403	·0025
26	29	32	39	·49	·29	·17	·05		
25	29	35	37	·57	·27	·10	·07	9·522	·0029
26	28	34	38	·49	·34	·12	·06		
24	32	33	37	·66	·15	·13	·06	9·549	·0034
26	30	31	39	·50	·25	·21	·05		
24	31	35	36	·66	·17	·09	·07	9·556	·0038
27	28	32	39	·42	·35	·18	·05		
25	30	33	38	·57	·23	·14	·06	9·576	·0040
25	29	36	36	·57	·27	·08	·08	9·602	·0043
27	27	34	38	·41	·41	·12	·06		
25	31	32	38	·58	·20	·17	·06	9·720	·0046
26	28	35	37	·49	·34	·10	·07	9·763	·0050
24	32	34	36	·66	·15	·11	·08	9·774	·0057
27	29	31	39	·43	·30	·22	·05		
26	28	36	36	·49	·34	·09	·09	9·842	·0061
27	27	35	37	·41	·41	·11	·07		
24	33	33	36	·66	·13	·13	·08	9·845	·0065
27	30	30	39	·43	·26	·26	·05		
24	32	35	35	·67	·15	·09	·09	9·848	·0069
28	28	31	39	·36	·36	·22	·05		
25	30	34	37	·58	·23	·12	·07	9·887	·0078
26	29	33	38	·50	·29	·15	·06		
27	27	36	36	·41	·41	·09	·09	9·920	·0079
24	33	34	35	·67	·13	·11	·09	9·991	·0090
28	29	30	39	·37	·31	·27	·05		
25	30	35	36	·58	·23	·10	·09	10·039	·0103
27	28	33	38	·42	·36	·16	·06		
24	34	34	34	·67	·11	·11	·11	10·062	·0105
29	29	29	39	·32	·32	·32	·05		
25	31	33	37	·58	·20	·15	·07	10·100	·0119
26	30	32	38	·50	·25	·18	·06		
25	32	32	37	·58	·17	·17	·07	10·170	·0127
26	31	31	38	·50	·22	·22	·06		
26	29	34	37	·50	·29	·13	·08	10·195	·0136
25	31	34	36	·58	·20	·13	·09	10·321	·0158
27	29	32	38	·43	·31	·19	·07		
26	29	35	36	·50	·30	·11	·09	10·346	·0181
27	28	34	37	·43	·36	·14	·08		
25	31	35	35	·58	·20	·11	·11	10·394	·0195
28	28	32	38	·37	·37	·19	·07		
25	32	33	36	·58	·18	·15	·09	10·460	·0225
27	30	31	38	·44	·27	·23	·07		
26	30	33	37	·50	·26	·16	·08	10·475	·0240
27	28	35	36	·43	·36	·12	·10	10·496	·0256
25	32	34	35	·58	·18	·13	·11	10·602	·0297
28	29	31	38	·38	·32	·23	·07		
26	31	32	37	·51	·22	·19	·08	10·612	·0317

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
					$n=7$ (*cont.*)				
25	33	33	35	·59	·15	·15	·11	10·670	·0341
28	30	30	38	·38	·27	·27	·07		
26	30	34	36	·51	·26	·14	·10	10·693	·0390
27	29	33	37	·44	·31	·17	·08		
25	33	34	34	·59	·15	·13	·13	10·740	·0417
29	29	30	38	·32	·32	·28	·07		
26	30	35	35	·51	·26	·12	·12	10·765	·0445
28	28	33	37	·37	·37	·17	·09		
26	31	33	36	·51	·22	·16	·10	10·897	·0521
27	30	32	37	·44	·27	·20	·09		
27	29	34	36	·44	·31	·14	·10	10·910	·0560
26	32	32	36	·51	·19	·19	·10	10·964	·0603
27	31	31	37	·44	·23	·23	·09		
27	29	35	35	·44	·32	·12	·12	10·981	·0648
28	28	34	36	·37	·37	·15	·11		
26	31	34	35	·51	·23	·14	·12	11·037	·0750
28	29	32	37	·38	·32	·20	·09		
28	28	35	35	·38	·38	·12	·12	11·052	·0763
26	32	33	35	·51	·20	·17	·12	11·171	·0898
28	30	31	37	·39	·28	·24	·09		
27	30	33	36	·44	·27	·17	·11	11·179	·0966
26	32	34	34	·51	·20	·14	·14	11·239	·1044
29	29	31	37	·33	·33	·24	·09		
26	33	33	34	·52	·17	·17	·15	11·305	·1134
29	30	30	37	·33	·29	·29	·09		
27	31	32	36	·45	·24	·20	·11	11·312	·1225
27	30	34	35	·45	·28	·15	·13	11·318	·1409
28	29	33	36	·38	·33	·18	·11		
28	29	34	35	·39	·33	·15	·13	11·456	·1532
27	31	33	35	·45	·24	·18	·13	11·515	·1813
28	30	32	36	·39	·29	·21	·11		
27	32	32	35	·45	·21	·21	·13	11·580	·1974
28	31	31	36	·39	·25	·25	·11		
27	31	34	34	·45	·24	·15	·15	11·582	·2136
29	29	32	36	·34	·34	·21	·11		
27	32	33	34	·45	·21	·18	·16	11·713	·2563
29	30	31	36	·34	·29	·25	·12		
28	30	33	35	·39	·29	·18	·14	11·717	·2778
27	33	33	33	·45	·18	·18	·18	11·778	·2860
30	30	30	36	·29	·29	·29	·12		
28	30	34	34	·39	·29	·16	·16	11·784	·3108
29	29	33	35	·34	·34	·19	·14		
28	31	32	35	·40	·25	·22	·14	11·846	·3392

$\Sigma_1 r$	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
					$n=7$ (*cont.*)				
29	29	34	34	·34	·34	·16	·16	11·850	·3464
28	31	33	34	·40	·25	·19	·16	11·977	·4214
29	30	32	35	·34	·30	·22	·14		
28	32	32	34	·40	·22	·22	·16	12·041	·4644
29	31	31	35	·35	·26	·26	·14		
28	32	33	33	·40	·22	·19	·19	12·105	·5138
30	30	31	35	·30	·30	·26	·14		
29	30	33	34	·35	·30	·19	·17	12·108	·5633
29	31	32	34	·35	·26	·22	·17	12·234	·6283
29	31	33	33	·35	·26	·19	·19	12·298	·7028
30	30	32	34	·30	·30	·23	·17		
29	32	32	33	·35	·23	·23	·20	12·361	·7880
30	31	31	34	·30	·26	·26	·17		
30	30	33	33	·30	·30	·20	·20	12·362	·8093
30	31	32	33	·31	·27	·23	·20	12·487	·9210
31	31	31	33	·27	·27	·27	·20	12·550	·9635
30	32	32	32	·31	·23	·23	·23		
31	31	32	32	·27	·27	·23	·23	12·612	1·0000
					$n=8$				
24	32	40	48	1	—	—	—	0	
24	32	41	47	1	—	—	—	1·309	
24	33	39	48	1	—	—	—		
25	31	40	48	·88	·12	—	—		
24	32	42	46	1	—	—	—	1·954	
24	34	38	48	1	—	—	—		
26	30	40	48	·75	·25	—	—		
24	32	43	45	1	—	—	—	2·298	
24	35	37	48	1	—	—	—		
27	29	40	48	·62	·38	—	—		
24	32	44	44	1	—	—	—	2·408	
24	36	36	48	1	—	—	—		
28	28	40	48	·50	·50	—	—		
25	31	41	47	·88	·12	—	—	2·618	
24	33	40	47	1	—	—	—	2·678	
25	32	39	48	·88	·11	·01	—		
25	33	39	47	·88	·09	·02	—	3·220	
26	30	41	47	·75	·25	—	—	3·263	
25	31	42	46	·88	·12	—	—		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n=8$ (*cont.*)					
24	33	41	46	1	—	—	—		
24	34	39	47	1	—	—	—	3·390	
25	33	38	48	·88	·09	·03	—		
26	31	39	48	·76	·22	·02	—		
25	31	43	45	·88	·12	—	—	3·608	
27	29	41	47	·62	·38	—	—		
25	31	44	44	·88	·12	—	—	3·717	
28	28	41	47	·50	·50	—	—		
24	33	42	45	1	—	—	—		
24	35	38	47	1	—	—	—	3·810	
25	34	37	48	·88	·08	·04	—		
27	30	39	48	·67	·32	·03	—		
26	30	42	46	·75	·25	—	—	3·908	
24	33	43	44	1	—	—	—		
24	36	37	47	1	—	—	—	4·008	
25	35	36	48	·88	·07	·05	—		
28	29	39	48	·54	·43	·03	—		
25	32	40	47	·88	·11	·01	—	4·047	
24	34	40	46	1	—	—	—		
26	32	38	48	·77	·19	·04	—	4·168	
26	30	43	45	·75	·25	—	—		
27	29	42	46	·62	·38	—	—	4·252	
26	30	44	44	·75	·25	—	—		
28	28	42	46	·50	·50	—	—	4·362	
27	29	43	45	·62	·38	—	—	4·597	
24	34	41	45	1	—	—	—		
24	35	39	46	1	—	—	—	4·660	
26	33	37	48	·77	·16	·07	—		
27	31	38	48	·66	·28	·06	—		
27	29	44	44	·62	·38	—	—	4·707	
28	28	43	45	·50	·50	—	—		
26	31	40	47	·76	·22	·02	—	4·757	
25	32	41	46	·88	·11	·01	—		
28	28	44	44	·50	·50	—	—	4·816	
24	34	42	44	1	—	—	—		
24	36	38	46	1	—	—	—	4·937	
26	34	36	48	·78	·13	·09	—		
28	30	38	48	·56	·37	·06	—		
24	34	43	43	1	—	—	—		
24	37	37	46	1	—	—	—	5·026	
26	35	35	48	·78	·11	·11	—		
29	29	38	48	·47	·47	·07	—		
25	32	42	45	·88	·11	·01	—	5·177	
27	30	40	47	·65	·32	·03	—		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n=8$ (*cont.*)					
24	35	40	45	1	—	—	—		
27	32	37	48	·67	·24	·09	—	5·222	
25	34	38	47	·88	·08	·03	—	5·308	
25	32	43	44	·88	·11	·01	·01		
28	29	40	47	·54	·43	·03	—	5·375	
26	31	41	46	·76	·22	·02	—	5·466	
24	35	41	44	1	—	—	—		
24	36	39	45	1	—	—	—	5·572	
27	33	36	48	·68	·21	·11	—		
28	31	37	48	·58	·32	·10	—		
25	35	37	47	·88	·07	·05	—	5·583	
25	33	40	46	·88	·10	·02	—	5·595	
26	32	39	47	·77	·19	·04	—		
25	36	36	47	·88	·06	·06	—	5·672	
24	35	42	43	1	—	—	—		
24	37	38	45	1	—	—	—	5·741	
27	34	35	48	·68	·17	·14	—		
29	30	37	48	·49	·45	·10	—		
27	30	41	46	·65	·32	·02	·01	5·886	
26	31	42	45	·76	·22	·01	·01		
24	36	40	44	1	—	—	—	5·994	
28	32	36	48	·59	·28	·13	—		
28	29	41	46	·54	·43	·02	·01		
26	31	43	44	·76	·22	·01	·01	6·084	
25	33	41	45	·88	·10	·02	·01		
27	31	39	47	·66	·28	·05	·01		
26	33	38	47	·77	·16	·06	·01	6·150	
25	34	39	46	·88	·08	·03	·01		
24	36	41	43	1	—	—	—		
24	37	39	44	1	—	—	—	6·236	
28	33	35	48	·60	·24	·16	—		
29	31	36	48	·51	·35	·14	—		
27	30	42	45	·65	·32	·01	·01	6·305	
24	36	42	42	1	—	—	—		
24	38	38	44	1	—	—	—	6·316	
28	34	34	48	·60	·20	·20	—		
30	30	36	48	·43	·43	·14	—		
28	30	39	47	·56	·37	·06	·01	6·360	
25	33	42	44	·88	·09	·01	·01		
26	32	40	46	·78	·18	·04	·01	6·374	
29	29	39	47	·47	·47	·06	·01	6·449	
25	33	43	43	·88	·09	·01	·01		
25	35	38	46	·88	·07	·04	·01	6·495	
26	34	37	47	·78	·14	·08	·01		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
$n=8$ (*cont.*)									
28	29	42	45	·54	·43	·02	·01	6·503	
27	30	43	44	·65	·32	·02	·01		
24	37	40	43	1	—	—	—	6·552	
29	32	35	48	·52	·30	·18	—		
26	35	36	47	·78	·12	·10	·01	6·661	
25	36	37	46	·88	·06	·05	·01		
28	29	43	44	·54	·43	·02	·01	6·701	
24	37	41	42	1	—	—	—	6·705	
24	38	39	43	1	—	—	—		
29	33	34	48	·52	·26	·22	—		
30	31	35	48	·44	·37	·18	—		
27	32	38	47	·67	·24	·08	·01	6·706	
25	34	40	45	·88	·08	·03	·01		
27	31	40	46	·66	·28	·04	·01	6·855	
26	32	41	45	·77	·19	·03	·01		
24	38	40	42	1	—	—	—	6·931	
30	32	34	48	·45	·32	·23	—		
26	33	39	46	·77	·16	·05	·01	6·985	
24	38	41	41	1	—	—	—	7·005	
24	39	39	42	1	—	—	—		
30	33	33	48	·45	·27	·27	—		
31	31	34	48	·38	·38	·23	—		
25	34	41	44	·88	·08	·02	·01	7·052	
28	31	38	47	·58	·32	·09	·01		
27	33	37	47	·68	·21	·10	·01	7·120	
25	35	39	45	·88	·07	·03	·01		
26	32	42	44	·78	·18	·03	·02	7·136	
28	30	40	46	·56	·38	·05	·01		
24	39	40	41	1	—	—	—	7·152	
31	32	33	48	·39	·33	·28	—		
29	29	40	46	·47	·47	·05	·01	7·218	
26	32	43	43	·77	·19	·02	·02		
25	34	42	43	·88	·08	·02	·02	7·220	
29	30	38	47	·49	·41	·09	·01		
24	40	40	40	1	—	—	—	7·225	
32	32	32	48	·33	·33	·33	—		
27	31	41	45	·67	·28	·04	·02	7·341	
27	34	36	47	·69	·18	·13	·01	7·357	
25	36	38	45	·88	·06	·04	·01		
26	34	38	46	·78	·14	·07	·01	7·396	
25	37	37	45	·88	·05	·05	·01	7·435	
27	35	35	47	·69	·15	·15	·01		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
$n=8$ (*cont.*)									
28	32	37	47	·59	·28	·12	·01	7·535	
25	35	40	44	·88	·07	·03	·01		
27	32	39	46	·68	·24	·07	·01	7·538	
26	33	40	45	·78	·16	·04	·02		
27	31	42	44	·67	·28	·03	·02	7·615	
28	30	41	45	·56	·37	·04	·02		
26	35	37	46	·78	·12	·09	·01	7·632	
29	29	41	45	·47	·47	·04	·02	7·704	
27	31	43	43	·67	·28	·03	·03		
26	36	36	46	·78	·10	·10	·01	7·709	
29	31	37	47	·51	·36	·12	·01	7·773	
25	35	41	43	·88	·07	·03	·02		
28	33	36	47	·60	·24	·15	·01	7·841	
25	36	39	44	·88	·06	·04	·02		
25	35	42	42	·88	·07	·02	·02	7·851	
30	30	37	47	·43	·43	·13	·01		
28	31	39	46	·58	·33	·08	·02	7·882	
26	33	41	44	·78	·17	·04	·02		
28	30	42	44	·56	·38	·04	·02	7·888	
29	29	42	44	·47	·47	·04	·02	7·977	
28	30	43	43	·57	·37	·03	·03		
28	34	35	47	·60	·21	·18	·01	7·990	
25	37	38	44	·89	·05	·05	·02		
26	34	39	45	·78	·14	·06	·02	8·012	
27	33	38	46	·68	·21	·09	·02		
26	33	42	43	·78	·17	·03	·03	8·048	
29	30	39	46	·49	·41	·08	·02		
29	29	43	43	·47	·47	·03	·03	8·066	
27	32	40	45	·68	·25	·06	·02	8·088	
29	32	36	47	·52	·31	·16	·01	8·149	
25	36	40	43	·88	·06	·03	·02		
30	31	36	47	·45	·38	·16	·01	8·299	
25	36	41	42	·88	·06	·03	·02		
27	34	37	46	·69	·19	·11	·02	8·315	
26	35	38	45	·78	·12	·07	·02		
29	33	35	47	·53	·27	·19	·01	8·367	
25	37	39	43	·89	·06	·04	·02		
26	34	40	44	·78	·14	·05	·02	8·423	
28	32	38	46	·60	·28	·10	·02		
28	31	40	45	·58	·33	·07	·02	8·430	
27	32	41	44	·68	·25	·05	·03		
29	34	34	47	·53	·23	·23	·01	8·439	
25	38	38	43	·89	·05	·05	·02		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
$n=8$ (*cont.*)									
27	35	36	46	·69	·16	·13	·02	8·462	
26	36	37	45	·78	·11	·09	·02		
25	37	40	42	·89	·06	·03	·02	8·586	
30	32	35	47	·46	·33	·20	·01		
27	32	42	43	·68	·25	·04	·03	8·595	
29	30	40	45	·49	·41	·07	·03		
27	33	39	45	·68	·22	·08	·02	8·624	
31	31	35	47	·39	·39	·20	·01	8·659	
25	37	41	41	·89	·06	·03	·03		
26	34	41	43	·78	·15	·04	·03	8·659	
29	31	38	46	·51	·36	·11	·02		
30	33	34	47	·46	·28	·24	·01	8·728	
25	38	39	42	·89	·05	·04	·03		
26	34	42	42	·78	·15	·04	·04	8·736	
30	30	38	46	·43	·43	·11	·02		
28	31	41	44	·58	·33	·06	·03	8·771	
26	35	39	44	·78	·13	·06	·03	8·791	
28	33	37	46	·60	·25	·13	·02		
25	38	40	41	·89	·05	·04	·03	8·869	
31	32	34	47	·40	·34	·25	·01		
28	31	42	43	·58	·33	·05	·04	8·936	
29	30	41	44	·50	·41	·06	·03		
25	39	39	41	·89	·04	·04	·03	8·938	
31	33	33	47	·40	·29	·29	·01		
27	34	38	45	·69	·19	·10	·03	8·989	
26	36	38	44	·78	·11	·08	·03	9·004	
28	34	36	46	·61	·22	·16	·02		
32	32	33	47	·35	·35	·30	·01	9·007	
25	39	40	40	·89	·04	·04	·04		
27	33	40	44	·68	·22	·07	03	9·031	
28	32	39	45	·60	·29	·09	·03		
28	35	35	46	·61	·18	·18	·02	9·075	
26	37	37	44	·78	·09	·09	·03		
29	32	37	46	·52	·31	·14	·02	9·093	
26	35	40	43	·78	·13	·06	·03		
29	30	42	43	·50	·41	·05	·04	9·100	
27	35	37	45	·69	·16	·12	·03	9·202	
30	31	37	46	·45	·38	·14	·02	9·241	
26	35	41	42	·78	·13	·05	·04		
27	33	41	43	·68	·22	·06	·04	9·265	
29	31	39	45	·51	·36	·10	·03		
27	36	36	45	·69	·14	·14	·03	9·272	
30	30	39	45	·44	·44	·10	·03	9·342	·0000
27	33	42	42	·68	·22	·05	·05		
29	33	36	46	·53	·27	·17	·02	9·373	·0001
26	36	39	43	·78	·11	·07	·04		
28	32	40	44	·60	·29	·08	·04	9·436	·0001
27	34	39	44	·69	·19	·09	·04	9·459	·0001
28	33	38	45	·60	·25	·11	·03		
26	37	38	43	·78	·10	·08	·04	9·510	·0001
29	34	35	46	·54	·24	·20	·02		
26	36	40	42	·78	·11	·06	·04	9·587	·0001
30	32	36	46	·46	·33	·18	·03		
26	36	41	41	·78	·11	·05	·05	9·658	·0001
31	31	36	46	·40	·40	·18	·03		
29	31	40	44	·51	·36	·08	·04	9·668	·0001
28	32	41	43	·60	·29	·07	·05		
28	34	37	45	·61	·22	·14	·03	9·735	·0001
27	35	38	44	·69	·17	·10	·04		
28	32	42	42	·60	·29	·06	·06	9·745	·0001
30	30	40	44	·44	·44	·09	·04		
29	32	38	45	·53	·32	·12	·03	9·758	·0001
27	34	40	43	·69	·19	·07	·04		
26	37	39	42	·78	·10	·07	·04	9·790	·0001
30	33	35	46	·47	·29	·21	·03		
26	38	38	42	·78	·08	·08	·05	9·857	·0001
30	34	34	46	·47	·25	·25	·03		
27	36	37	44	·69	·14	·12	·04	9·870	·0002
28	35	36	45	·61	·19	·16	·03		
29	31	41	43	·52	·36	·07	·05	9·901	·0002
30	31	38	45	·45	·38	·13	·04	9·904	·0002
27	34	41	42	·69	·19	·06	·05		
28	33	39	44	·61	·25	·10	·04	9·925	·0002
31	32	35	46	·41	·35	·22	·03	9·928	·0002
26	37	40	41	·79	·10	·06	·05		
29	31	42	42	·52	·36	·06	·06	9·977	·0002
30	30	41	43	·44	·44	·07	·05		
30	30	42	42	·44	·44	·06	·06	10·053	·0002
26	38	39	41	·78	·09	·07	·05	10·060	·0002
31	33	34	46	·41	·30	·26	·03		
29	33	37	45	·53	·28	·15	·04	10·097	·0003
27	35	39	43	·69	·17	·09	·05		
26	38	40	40	·79	·09	·06	·06	10·127	·0003
32	32	34	46	·35	·35	·26	·03		
26	39	39	40	·79	·07	·07	·06	10·192	·0003
32	33	33	46	·36	·31	·31	·03		
28	33	40	43	·61	·26	·09	·05	10·221	·0003
29	32	39	44	·53	·32	·11	·05		
28	34	38	44	·61	·22	·12	·04	10·261	·0003
27	36	38	43	·69	·15	·11	·05	10·296	·0004
29	34	36	45	·54	·24	·18	·04		
30	32	37	45	·47	·34	·16	·04	10·308	·0004
27	35	40	42	·69	·17	·08	·06		
27	37	37	43	·70	·13	·13	·05	10·361	·0004
29	35	35	45	·54	·21	·21	·04		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
					$n=8$ (*cont.*)				
30	31	39	44	·46	·39	·11	·05	10·366	·0005
28	33	41	42	·61	·26	·07	·06		
27	35	41	41	·69	·17	·07	·07	10·378	·0005
31	31	37	45	·40	·40	·16	·04		
28	35	37	44	·61	·20	·14	·05	10·458	·0005
29	32	40	43	·53	·32	·09	·06	10·516	·0005
27	36	39	42	·70	·15	·10	·06	10·570	·0006
30	33	36	45	·47	·30	·19	·04		
28	34	39	43	·61	·23	·11	·06	10·619	·0007
29	33	38	44	·54	·28	·13	·05		
29	32	41	42	·53	·32	·08	·07	10·660	·0008
30	31	40	43	·46	·39	·10	·06		
30	34	35	45	·48	·26	·22	·04	10·699	·0008
27	37	38	42	·70	·13	·11	·06		
27	36	40	41	·70	·15	·08	·07	10·704	·0009
31	32	36	45	·41	·35	·19	·04		
28	36	36	44	·62	·17	·17	·05	10·723	·0009
30	31	41	42	·46	·39	·08	·07	10·804	·0010
28	34	40	42	·61	·23	·09	·07	10·828	·0011
30	32	38	44	·47	·34	·14	·05		
28	35	38	43	·62	·20	·13	·06	10·877	·0012
29	34	37	44	·54	·25	·16	·05		
31	31	38	44	·40	·40	·14	·05	10·896	·0013
28	34	41	41	·61	·23	·08	·08		
31	33	35	45	·42	·31	·23	·04	10·897	·0014
27	37	39	41	·70	·13	·10	·07		
27	38	38	41	·70	·11	·11	·07	10·960	·0015
31	34	34	45	·42	·27	·27	·04		
27	37	40	40	·70	·13	·08	·08	10·962	·0016
32	32	35	45	·36	·36	·23	·04		
29	33	39	43	·54	·28	·12	·06	10·976	·0016
29	35	36	44	·55	·22	·19	·05	11·004	·0018
28	36	37	43	·62	·17	·15	·06		
27	38	39	40	·70	·11	·10	·09	11·089	·0020
32	33	34	45	·37	·32	·27	·04		
28	35	39	42	·62	·20	·11	·07	11·147	·0022
30	33	37	44	·48	·30	·17	·05		
27	39	39	39	·70	·10	·10	·10	11·152	·0023
33	33	33	45	·32	·32	·32	·05		
29	33	40	42	·54	·28	·10	·07	11·182	·0025
30	32	39	43	·47	·34	·12	·06		
31	31	39	43	·40	·40	·12	·07	11·250	·0026
29	33	41	41	·54	·28	·09	·09		
28	35	40	41	·62	·20	·10	·08	11·280	·0029
31	32	37	44	·41	·36	·17	·06		
29	34	38	43	·55	·25	·14	·06	11·291	·0031
28	36	38	42	·62	·18	·13	·07	11·335	·0034
30	34	36	44	·48	·26	·20	·06		
30	32	40	42	·47	·34	·11	·08	11·387	·0036
30	35	35	44	·48	·23	·23	·06	11·397	·0037
28	37	37	42	·62	·15	·15	·07		
31	31	40	42	·41	·41	·11	·08	11·455	·0039
30	32	41	41	·47	·34	·09	·09		
29	35	37	43	·55	·22	·17	·07	11·478	·0042
31	31	41	41	·41	·41	·09	·09	11·523	·0042
31	33	36	44	·42	·31	·21	·06	11·529	·0047
28	36	39	41	·62	·18	·12	·09		
29	36	36	43	·55	·19	·19	·07	11·540	·0048
30	33	38	43	·48	·30	·15	·07	11·559	·0054
29	34	39	42	·55	·25	·12	·08		
28	36	40	40	·62	·18	·10	·10	11·594	·0056
32	32	36	44	·37	·37	·21	·06		
28	37	38	41	·62	·16	·13	·09	11·652	·0063
31	34	35	44	·43	·28	·24	·06		
29	34	40	41	·55	·25	·11	·09	11·690	·0070
31	32	38	43	·42	·36	·15	·07		
32	33	35	44	·37	·32	·24	·06	11·778	·0078
28	37	39	40	·62	·16	·12	·10		
29	35	38	42	·55	·22	·15	·08	11·805	·0087
30	34	37	43	·48	·27	·18	·07		
30	33	39	42	·48	·31	·13	·08	11·825	·0091
32	34	34	44	·37	·28	·28	·06	11·839	·0096
28	38	38	40	·62	·14	·14	·10		
28	38	39	39	·63	·14	·12	·12	11·901	·0101
33	33	34	44	·33	·33	·28	·06		
29	36	37	42	·55	·20	·17	·08	11·926	·0113
30	35	36	43	·49	·24	·20	·07		
31	32	39	42	·42	·36	·13	·09	11·955	·0125
30	33	40	41	·48	·31	·11	·10		
29	35	39	41	·55	·22	·13	·10	11·996	·0138
31	33	37	43	·43	·32	·18	·07		
29	35	40	40	·55	·22	·11	·11	12·059	·0146
32	32	37	43	·37	·37	·18	·07		
31	32	40	41	·42	·36	·12	·10	12·085	·0154
30	34	38	42	·49	·27	·16	·09	12·128	·0163
29	36	38	41	·55	·20	·15	·10	12·177	·0182
31	34	36	43	·43	·28	·21	·08		
29	37	37	41	·56	·17	·17	·10	12·236	·0193
31	35	35	43	·43	·25	·25	·08		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n=8$ (*cont.*)					
29	36	39	40	·56	·20	·13	·11	12·300	·0218
32	33	36	43	·38	·33	·22	·08		
30	35	37	42	·49	·24	·18	·09	12·307	·0231
31	33	38	42	·43	·32	·16	·09	12·318	·0257
30	34	39	41	·49	·27	·14	·10		
30	36	36	42	·49	·21	·21	·09	12·367	·0264
30	34	40	40	·49	·27	·12	·12	12·381	·0279
32	32	38	42	·37	·37	·16	·09		
29	37	38	40	·56	·17	·15	·12	12·419	·0312
32	34	35	43	·38	·29	·25	·08		
33	33	35	43	·33	·33	·25	·08	12·480	·0331
29	37	39	39	·56	·18	·13	·13		
31	33	39	41	·43	·32	·14	·11	12·507	·0350
29	38	38	39	·56	·15	·15	·13	12·539	·0371
33	34	34	43	·34	·29	·29	·08		
30	35	38	41	·49	·24	·16	·11	12·555	·0415
31	34	37	42	·43	·28	·19	·09		
31	33	40	40	·43	·32	·12	·12	12·569	·0437
32	32	39	41	·37	·37	·14	·11		
32	32	40	40	·38	·38	·12	·12	12·632	·0444
30	36	37	41	·50	·21	·19	·11	12·672	·0500
31	35	36	42	·44	·25	·22	·09		
32	33	37	42	·38	·33	·19	·09	12·677	·0556
30	35	39	40	·49	·24	·14	·12		
33	33	39	39	·34	·34	·16	·16	12·683	·0593
31	34	38	41	·44	·29	·17	·11	12·801	·0630
30	36	38	40	·50	·21	·16	·12	12·853	·0713
32	34	36	42	·39	·29	·22	·10		
32	35	35	42	·39	·26	·26	·10	12·910	·0759
30	37	37	40	·50	·19	·19	·13		
30	36	39	39	·50	·21	·14	·14	12·912	·0806
33	33	36	42	·34	·34	·23	·10		
31	34	39	40	·44	·29	·15	·13	12·922	·0902
32	33	38	41	·38	·33	·17	·11		
31	35	37	41	·44	·25	·19	·11	12·975	·0955
33	34	35	42	·34	·30	·26	·10	13·028	·1076
30	37	38	39	·50	·19	·17	·15		
32	33	39	40	·38	·33	·15	·13	13·043	·1137
30	38	38	38	·50	·17	·17	·17	13·086	·1160
34	34	34	42	·30	·30	·30	·10		
31	35	38	40	·44	·26	·17	·13	13·154	·1317
32	34	37	41	·39	·30	·20	·12		
33	33	37	41	·34	·34	·20	·12	13·212	·1407
31	35	39	39	·44	·26	·15	·15		
31	36	37	40	·44	·23	·20	·13	13·268	·1608
32	35	36	41	·39	·26	·23	·12		
32	34	38	40	·39	·30	·18	·13	13·331	·1723
31	36	38	39	·45	·23	·17	·15	13·384	·1981
33	34	36	41	·35	·30	·23	·12		
33	33	38	40	·34	·34	·18	·14	13·390	·2111
32	34	39	39	·39	·30	·15	·15		
31	37	37	39	·45	·20	·20	·15	13·440	·2257
33	35	35	41	·35	·27	·27	·12		
34	34	35	41	·31	·31	·27	·12	13·498	·2422
31	37	38	38	·45	·20	·18	·18		
32	35	37	40	·39	·26	·20	·14	13·501	·2588
32	36	36	40	·40	·23	·23	·14	13·557	·2681
32	35	38	39	·40	·27	·18	·16	13·616	·3107
33	34	37	40	·35	·31	·21	·14		
32	36	37	39	·40	·24	·21	·16	13·728	·3648
33	35	36	40	·35	·27	·24	·14		
33	34	38	39	·35	·31	·18	·16	13·731	·3920
32	36	38	38	·40	·24	·18	·18	13·785	·4226
34	34	36	40	·31	·31	·24	·14		
32	37	37	38	·40	·21	·21	·18	13·840	·4571
34	35	35	40	·31	·27	·27	·14		
33	35	37	39	·35	·27	·21	·16	13·898	·4962
33	36	36	39	·35	·24	·24	·16	13·954	·5182
34	34	37	39	·31	·31	·21	·16	13·954	·5623
33	35	38	38	·35	·27	·19	·19		
34	34	38	38	·31	·31	·19	·19	14·011	·5747
31	36	36	41	·44	·22	·22	·11	14·038	·5777
33	36	37	38	·36	·24	·21	·19	14·065	·6897
34	35	36	39	·31	·28	·24	·17		
35	35	35	39	·28	·28	·28	·17	14·120	·7107
33	37	37	37	·36	·21	·21	·21		
34	35	37	38	·32	·28	·22	·19	14·176	·7819
34	36	36	38	·32	·25	·25	·19	14·231	·8220
35	35	36	38	·28	·28	·25	·19	14·286	·9122
34	36	37	37	·32	·25	·22	·22		
35	35	37	37	·28	·28	·22	·22	14·340	·9375
35	36	36	37	·28	·25	·25	·22	14·368	·9946
36	36	36	36	·25	·25	·25	·25	14·450	1·0000

5 treatments. (*Design*: $t = 5, \lambda = 1, b = 10, r = 4, k = 2$)

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
							$n=1$				
4	5	6	7	8	1	—	—	—	—	0	·1172
4	5	7	7	7	1	—	—	—	—	·903	·2344
4	6	6	6	8	1	—	—	—	—		
5	5	5	7	8	·33	·33	·33	—	—		
4	6	6	7	7	1	—	—	—	—	1·579	·4688
5	5	6	6	8	·38	·38	·12	·12	—		
5	5	6	7	7	·38	·38	·14	·05	·05	2·235	·7031
5	6	6	6	7	·42	·17	·17	·17	·07	2·640	·9766
6	6	6	6	6	·20	·20	·20	·20	·20	3·010	1·0000
							$n=2$				
8	10	12	14	16	1	—	—	—	—	·000	·0001
8	10	12	15	15	1	—	—	—	—	·602	·0006
8	10	13	13	16	1	—	—	—	—		
8	11	11	14	16	1	—	—	—	—		
9	9	12	14	16	·50	·50	—	—	—		
8	11	11	15	15	1	—	—	—	—	1·204	·0009
9	9	12	15	15	·50	·50	—	—	—		
9	9	13	13	16	·50	·50	—	—	—		
8	10	13	14	15	1	—	—	—	—	1·498	·0030
8	11	12	13	16	1	—	—	—	—		
9	10	11	14	16	·59	·28	·13	—	—		
8	10	14	14	14	1	—	—	—	—	1·806	·0035
8	12	12	12	16	1	—	—	—	—		
10	10	10	14	16	·33	·33	·33	—	—		
9	9	13	14	15	·50	·50	—	—	—	2·100	·0048
9	10	11	15	15	·59	·28	·13	—	—		
8	11	12	14	15	1	—	—	—	—	2·359	·0090
9	10	12	13	16	·60	·29	·08	·04	—		
9	9	14	14	14	·50	·50	—	—	—	2·408	·0093
10	10	10	15	15	·33	·33	·33	—	—		
8	11	13	13	15	1	—	—	—	—	2·631	·0126
9	11	11	13	16	·62	·17	·17	·05	—		
8	11	13	14	14	1	—	—	—	—	2·898	·0227
8	12	12	13	15	1	—	—	—	—		
9	11	12	12	16	·62	·18	·10	·10	—		
10	10	11	13	16	·37	·37	·21	·06	—		
8	12	12	14	14	1	—	—	—	—	3·158	·0266
10	10	12	12	16	·38	·38	·12	·12	—		
9	10	12	14	15	·60	·29	·08	·02	·01	3·217	·0328
8	12	13	13	14	1	—	—	—	—	3·389	·0444
10	11	11	12	16	·40	·23	·23	·14	—		
9	10	13	13	15	·60	·29	·05	·05	·01	3·482	·0540
9	11	11	14	15	·62	·17	·17	·03	·01		
8	13	13	13	13	1	—	—	—	—	3·612	·0555
11	11	11	11	16	·25	·25	·25	·25	—		
9	10	13	14	14	·60	·29	·05	·03	·03	3·747	·0706
10	10	11	14	15	·37	·37	·21	·04	·02		
9	11	12	13	15	·62	·18	·11	·06	·02	3·966	·0926
9	12	12	12	15	·63	·12	·12	·12	·02	4·149	·0980
10	10	12	13	15	·38	·38	·14	·08	·02	4·219	·1319
9	11	12	14	14	·62	·19	·11	·04	·04		
9	11	13	13	14	·63	·19	·07	·07	·04	4·436	·1813
10	11	11	13	15	·40	·24	·24	·09	·03		
10	10	12	14	14	·38	·38	·14	·05	·05	4·471	·1948
10	11	12	12	15	·41	·25	·16	·16	·03	4·644	·2671
9	12	12	13	14	·63	·12	·12	·08	·05		
10	10	13	13	14	·38	·38	·09	·09	·05	4·684	·3053
10	11	11	14	14	·40	·24	·24	·06	·06		
9	12	13	13	13	·64	·13	·08	·08	·08	4·848	·3403
11	11	11	12	15	·27	·27	·27	·17	·03		
10	11	12	13	14	·41	·26	·16	·10	·06	5·049	·4989
10	12	12	12	14	·42	·17	·17	·17	·07	5·280	·5364
10	11	13	13	13	·42	·26	·11	·11	·11	5·288	·6123
11	11	11	13	14	·27	·27	·27	·11	·07		
10	12	12	13	13	·43	·17	·17	·11	·11	5·474	·7727
11	11	12	12	14	·28	·28	·18	·18	·07		
11	11	12	13	13	·28	·28	·19	·12	·12	5·665	·8866
11	12	12	12	13	·29	·19	·19	·19	·13	5·844	·9928
12	12	12	12	12	·20	·20	·20	·20	·20	6·021	1·0000
							$n=3$				
12	15	18	21	24	1	—	—	—	—	0	
12	15	18	22	23	1	—	—	—	—	·829	
12	15	19	20	24	1	—	—	—	—		
12	16	17	21	24	1	—	—	—	—		
13	14	18	21	24	·67	·33	—	—	—		
12	16	17	22	23	1	—	—	—	—	1·659	
13	14	18	22	23	·67	·33	—	—	—		
13	14	19	20	24	·67	·33	—	—	—		
12	15	19	21	23	1	—	—	—	—	1·840	
12	16	18	20	24	1	—	—	—	—		
13	15	17	21	24	·70	·22	·07	—	—		
12	15	19	22	22	1	—	—	—	—	2·077	
12	15	20	20	23	1	—	—	—	—		
12	16	19	19	24	1	—	—	—	—		
12	17	17	20	24	1	—	—	—	—		
13	16	16	21	24	·71	·14	·14	—	—		
14	14	17	21	24	·45	·45	·09	—	—		
12	15	20	21	22	1	—	—	—	—	2·511	
12	17	18	19	24	1	—	—	—	—		
14	15	16	21	24	·50	·31	·19	—	—		
13	14	19	21	23	·67	·33	—	—	—	2·670	
13	15	17	22	23	·70	·22	·07	—	—		
12	15	21	21	21	1	—	—	—	—	2·709	
12	18	18	18	24	1	—	—	—	—		
15	15	15	21	24	·33	·33	·33	—	—		
12	16	18	21	23	1	—	—	—	—	2·836	·0000
13	15	18	20	24	·71	·23	·05	·02	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=3$ (*cont.*)						
13	14	19	22	22	·67	·33	—	—	—		
13	14	20	20	23	·67	·33	—	—	—	2·906	·0001
13	16	16	22	23	·71	·14	·14	—	—		
14	14	17	22	23	·45	·45	·09	—	—		
12	16	18	22	22	1	—	—	—	—		
12	17	17	21	23	1	—	—	—	—	3·069	·0001
13	15	19	19	24	·71	·23	·03	·03	—		
14	14	18	20	24	·48	·46	·06	·02	—		
12	16	19	20	23	1	—	—	—	—		
13	16	17	20	24	·72	·16	·10	·02	—	3·248	·0001
12	17	17	22	22	1	—	—	—	—		
14	14	19	19	24	·46	·46	·04	·04	—	3·301	·0001
13	14	20	21	22	·67	·33	—	—	—		
14	15	16	22	23	·50	·31	·19	—	—	3·340	·0001
13	14	21	21	21	·67	·33	—	—	—		
15	15	15	22	23	·33	·33	·33	—	—	3·539	·0001
12	16	19	21	22	1	—	—	—	—		
12	17	18	20	23	1	—	—	—	—		
13	16	18	19	24	·72	·16	·07	·05	—	3·659	·0002
14	15	17	20	24	·50	·32	·14	·03	—		
13	15	18	21	23	·71	·23	·05	·01	—	3·832	·0003
12	16	20	20	22	1	—	—	—	—		
12	17	19	19	23	1	—	—	—	—		
13	17	17	19	24	·72	·11	·11	·05	—	3·837	·0003
14	16	16	20	24	·52	·22	·22	·03	—		
12	16	20	21	21	1	—	—	—	—		
12	18	18	19	23	1	—	—	—	—		
13	17	18	18	24	·73	·12	·08	·08	—	4·016	·0004
15	15	16	20	24	·36	·36	·24	·04	—		
12	17	18	21	22	1	—	—	—	—		
14	15	18	19	24	·51	·33	·10	·06	—	4·060	·0005
13	15	18	22	22	·71	·23	·05	·01	·01		
14	14	18	21	23	·46	·46	·07	·01	—	4·065	·0005
13	16	17	21	23	·72	·16	·10	·02	·01		
13	15	19	20	23	·71	·23	·04	·02	—	4·239	·0006
14	14	18	22	22	·46	·46	·07	·01	·01	4·298	·0006
12	17	19	20	22	1	—	—	—	—		
14	16	17	19	24	·53	·24	·16	·07	—	4·397	·0008
13	16	17	22	22	·72	·16	·10	·01	·01		
14	14	19	20	23	·46	·46	·05	·03	·01	4·470	·0009
12	17	19	21	21	1	—	—	—	—		
12	18	18	20	22	1	—	—	—	—		
14	16	18	18	24	·53	·24	·11	·11	—	4·568	·0011
15	15	17	19	24	·37	·37	·18	·08	—		
13	15	19	21	22	·71	·23	·04	·02	·01		
14	15	17	21	23	·51	·32	·14	·02	·01	4·648	·0013
12	17	20	20	21	1	—	—	—	—		
12	18	19	19	22	1	—	—	—	—		
14	17	17	18	24	·54	·17	·17	·12	—	4·727	·0016
15	16	16	19	24	·38	·27	·27	·09	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=3$ (*cont.*)						
12	18	18	21	21	1	—	—	—	—		
15	15	18	18	24	·38	·38	·12	·12	—	4·737	·0017
13	16	18	20	23	·72	·16	·07	·03	·01		
13	15	20	20	22	·71	·23	·03	·03	·01	4·801	·0019
14	16	16	21	23	·52	·22	·22	·02	·01	4·825	·0020
14	14	19	21	22	·46	·46	·05	·02	·01		
14	15	17	22	22	·51	·32	·14	·01	·01	4·879	·0021
13	16	19	19	23	·72	·17	·05	·05	·01		
13	17	17	20	23	·73	·12	·12	·04	·01	4·972	·0023
13	15	20	21	21	·71	·23	·03	·02	·02		
15	15	16	21	23	·36	·36	·24	·03	·01	5·003	·0025
12	18	19	20	21	1	—	—	—	—		
15	16	17	18	24	·39	·28	·19	·14	—	5·046	·0031
14	14	20	20	22	·46	·46	·03	·03	·01		
14	16	16	22	22	·52	·22	·22	·02	·02	5·056	·0032
12	18	20	20	20	1	—	—	—	—		
12	19	19	19	21	1	—	—	—	—		
15	17	17	17	24	·40	·20	·20	·20	—	5·197	·0035
16	16	16	18	24	·29	·29	·29	·14	—		
14	15	18	20	23	·51	·33	·10	·05	·01		
13	16	18	21	22	·72	·17	·07	·02	·02	5·200	·0040
14	14	20	21	21	·46	·46	·03	·02	·02		
15	15	16	22	22	·36	·36	·24	·02	·02	5·233	·0041
13	17	18	19	23	·73	·12	·08	·06	·01	5·299	·0044
12	19	19	20	20	1	—	—	—	—		
16	16	17	17	24	·29	·29	·21	·21	—	5·346	·0047
13	17	17	21	22	·73	·12	·12	·03	·02		
14	15	19	19	23	·51	·33	·07	·07	·01	5·368	·0050
13	18	18	18	23	·73	·09	·09	·09	·01	5·456	·0051
14	16	17	20	23	·53	·24	·17	·05	·01		
13	16	19	20	22	·72	·17	·06	·04	·02	5·526	·0060
14	15	18	21	22	·51	·33	·10	·03	·02	5·598	·0066
13	16	19	21	21	·72	·17	·06	·03	·03		
15	15	17	20	23	·37	·37	·18	·06	·01	5·695	·0072
14	16	18	19	23	·53	·24	·12	·09	·01		
13	17	18	20	22	·73	·12	·09	·04	·02	5·841	·0089
13	16	20	20	21	·72	·17	·04	·04	·03		
15	16	16	20	23	·38	·27	·27	·06	·01	5·850	·0098
14	15	19	20	22	·51	·33	·08	·05	·02		
14	16	17	21	22	·53	·24	·17	·04	·03	5·920	·0117
13	17	19	19	22	·73	·12	·06	·06	·02		
14	17	17	19	23	·54	·18	·18	·09	·02	5·991	·0128
13	17	18	21	21	·73	·12	·09	·03	·03		
15	15	18	19	23	·38	·38	·13	·09	·02	6·007	·0140
14	15	19	21	21	·51	·33	·08	·04	·04		
15	15	17	21	22	·37	·37	·18	·04	·03	6·088	·0153
14	17	18	18	23	·54	·18	·13	·13	·02		
13	18	18	19	22	·73	·09	·09	·07	·02	6·137	·0168
15	16	16	21	22	·39	·27	·27	·05	·03		
14	15	20	20	21	·51	·34	·06	·06	·04	6·242	·0186
13	17	19	20	21	·73	·13	·06	·05	·03		
15	16	17	19	23	·40	·28	·20	·10	·02	6·302	·0227

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=3$ (*cont.*)						
14	16	18	20	22	·53	·25	·13	·06	·03	6·378	·0250
13	18	18	20	21	·73	·09	·09	·05	·03	6·445	·0277
15	16	18	18	23	·40	·28	·15	·15	·02		
13	17	20	20	20	·73	·13	·05	·05	·05	6·448	·0286
16	16	16	19	23	·29	·29	·29	·11	·02		
14	16	19	19	22	·54	·25	·09	·09	·03	6·523	·0316
14	17	17	20	22	·54	·18	·18	·07	·03		
15	15	18	20	22	·38	·38	·14	·07	·03	6·542	·0348
14	16	18	21	21	·53	·25	·13	·05	·05		
13	18	19	19	21	·72	·09	·07	·07	·04	6·585	·0383
15	17	17	18	23	·41	·21	·21	·15	·02		
14	17	17	21	21	·54	·18	·18	·05	·05	6·686	·0404
15	15	19	19	22	·38	·38	·10	·10	·03		
15	15	18	21	21	·38	·38	·14	·05	·05	6·706	·0415
13	18	19	20	20	·73	·10	·07	·05	·05	6·727	·0462
16	16	17	18	23	·30	·30	·22	·16	·02		
14	17	18	19	22	·55	·19	·14	·10	·03	6·804	·0514
14	16	19	20	21	·54	·25	·10	·07	·05	6·830	·0624
15	16	17	20	22	·40	·28	·20	·08	·04		
13	19	19	19	20	·73	·07	·07	·07	·05	6·863	·0644
16	17	17	17	23	·31	·22	·22	·22	·02		
15	18	18	18	22	·55	·14	·14	·14	·04	6·941	·0650
14	16	20	20	20	·54	·25	·07	·07	·07	6·974	·0680
16	16	16	20	22	·29	·29	·29	·08	·04		
15	15	19	20	21	·38	·38	·10	·08	·05	6·992	·0755
15	16	17	21	21	·40	·28	·21	·06	·06		
15	16	18	19	22	·40	·29	·15	·11	·04	7·106	·0942
14	17	18	20	21	·55	·19	·14	·07	·05		
15	15	20	20	20	·38	·38	·08	·08	·08	7·134	·0958
16	16	16	21	21	·29	·29	·29	·06	·06		
14	17	19	19	21	·55	·19	·10	·10	·05	7·240	·1081
15	17	17	19	22	·41	·22	·22	·12	·04		
14	18	18	19	21	·55	·14	·14	·11	·06	7·372	·1239
15	17	18	18	22	·42	·22	·16	·16	·04		
14	17	19	20	20	·55	·19	·10	·08	·08	7·379	·1399
16	16	17	19	22	·31	·31	·22	·12	·04		
15	16	18	20	21	·41	·29	·16	·08	·06	7·404	·1566
14	18	18	20	20	·55	·14	·14	·08	·08	7·509	·1669
16	16	18	18	22	·31	·31	·17	·17	·04		
15	17	17	20	21	·41	·22	·22	·09	·06	7·537	·1885
15	16	19	19	21	·41	·29	·12	·12	·06		
14	18	19	19	20	·55	·15	·11	·11	·08	7·638	·2153
16	17	17	18	22	·31	·23	·23	·17	·05		
15	16	19	20	20	·41	·29	·12	·09	·09	7·674	·2436
16	16	17	20	21	·31	·31	·23	·09	·07		
17	17	17	17	22	·24	·24	·24	·24	·05	7·767	·2464
14	19	19	19	19	·56	·11	·11	·11	·11		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=3$ (*cont.*)						
15	17	18	19	21	·42	·22	·17	·12	·07	7·795	·2825
15	18	18	18	21	·42	·17	·17	·17	·07	7·921	·2902
15	17	18	20	20	·42	·22	·17	·09	·09	7·929	·3371
16	16	18	19	21	·31	·31	·17	·13	·07		
15	17	19	19	20	·42	·23	·13	·13	·10	8·056	·3974
16	17	17	19	21	·32	·24	·24	·13	·07		
16	16	18	20	20	·31	·31	·18	·10	·10	8·063	·4127
15	18	18	19	20	·43	·17	·17	·13	·10	8·179	·4898
16	17	18	18	21	·32	·24	·18	·18	·07		
16	17	17	20	20	·32	·24	·24	·10	·10	8·189	·5290
16	16	19	19	20	·32	·32	·13	·13	·10		
17	17	17	18	21	·25	·25	·25	·19	·08	8·304	·5619
15	18	19	19	19	·43	·18	·13	·13	·13		
16	17	18	19	20	·33	·24	·18	·14	·10	8·435	·6898
16	18	18	18	20	·33	·19	·19	·19	·11	8·555	·7169
16	17	19	19	19	·33	·25	·14	·14	·14	8·557	·7713
17	17	17	19	20	·25	·25	·25	·14	·11		
16	18	18	19	19	·33	·19	·19	·14	·14	8·676	·8749
17	17	18	18	20	·25	·25	·19	·19	·11		
17	17	18	19	19	·26	·26	·19	·15	·15	8·809	·9409
17	18	18	18	19	·26	·20	·20	·20	·15	8·914	·9965
18	18	18	18	18	·20	·20	·20	·20	·20	9·031	1·0000
					$n=4$						
16	20	24	28	32	1	—	—	—	—	0	
16	20	24	29	31	1	—	—	—	—		
16	20	25	27	32	1	—	—	—	—	·977	
16	21	23	28	32	1	—	—	—	—		
17	19	24	28	32	·75	·25	—	—	—		
18	18	24	28	32	·50	·50	—	—	—	1·204	
16	20	24	30	30	1	—	—	—	—		
16	20	26	26	32	1	—	—	—	—		
16	22	22	28	32	1	—	—	—	—		
16	21	23	29	31	1	—	—	—	—	1·954	
17	19	24	29	31	·75	·25	—	—	—		
17	19	25	27	32	·75	·25	—	—	—		
16	20	25	28	31	1	—	—	—	—	2·084	
16	21	24	27	32	1	—	—	—	—		
17	20	23	28	32	·77	·19	·04	—	—		
16	21	23	30	30	1	—	—	—	—		
16	22	22	29	31	1	—	—	—	—		
17	19	24	30	30	·75	·25	—	—	—	2·181	
17	19	26	26	32	·75	·25	—	—	—		
18	18	24	29	31	·50	·50	—	—	—		
18	18	25	27	32	·50	·50	—	—	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=4$ (*cont.*)						
16	22	22	30	30	1	—	—	—	—	2·408	·2825
18	18	24	30	30	·50	·50	—	—	—		·2902
18	18	26	26	32	·50	·50	—	—	—		·3371
16	20	25	29	30	1	—	—	—	—	2·468	·3974
16	20	26	27	31	1	—	—	—	—		
16	21	25	26	32	1	—	—	—	—		
16	22	23	27	32	1	—	—	—	—		
17	21	22	28	32	·78	·13	·09	—	—		·4127
18	19	23	28	32	·56	·37	·06	—	—		·4898
16	20	26	28	30	1	—	—	—	—	2·997	·5290
16	22	24	26	32	1	—	—	—	—		
18	20	22	28	32	·59	·28	·13	—	—		
17	19	25	28	31	·75	·25	—	—	—	3·061	·5619
17	20	23	29	31	·77	·19	·04	—	—		·6898
16	20	26	29	29	1	—	—	—	—	3·158	·7169
16	20	27	27	30	1	—	—	—	—		·7713
16	22	25	25	32	1	—	—	—	—		
16	23	23	26	32	1	—	—	—	—		
18	21	21	28	32	·60	·20	·20	—	—		
19	19	22	28	32	·43	·43	·14	—	—		·8749
16	21	24	28	31	1	—	—	—	—	3·187	·9409
17	20	24	27	32	·78	·18	·04	·01	—		·9965
17	20	23	30	30	·77	·19	·04	—	—	3·288	1·0000
18	18	25	28	31	·50	·50	—	—	—		
17	19	26	27	31	·75	·25	—	—	—	3·445	
17	21	22	29	31	·78	·13	·09	—	—		
18	19	23	29	31	·56	·37	·06	—	—		
17	19	25	29	30	·75	·25	—	—	—		
16	20	27	28	29	1	—	—	—	—	3·466	
16	23	24	25	32	1	—	—	—	—		
19	20	21	28	32	·45	·32	·23	—	—		
16	21	24	29	30	1	—	—	—	—	3·568	
16	22	23	28	31	1	—	—	—	—		
17	20	25	26	32	·78	·18	·03	·02	—		
18	19	24	27	32	·56	·38	·05	·01	—		
16	20	28	28	28	1	—	—	—	—	3·612	
16	24	24	24	32	1	—	—	—	—		
20	20	20	28	32	·33	·33	·33	—	—		
17	21	22	30	30	·78	·13	·09	—	—	3·672	
18	18	25	29	30	·50	·50	—	—	—		
18	18	26	27	31	·50	·50	—	—	—		
18	19	23	30	30	·56	·37	·06	—	—		
16	21	25	27	31	1	—	—	—	—	3·698	
17	21	23	27	32	·78	·14	·07	·01	—		
16	21	26	26	31	1	—	—	—	—	3·854	
17	22	22	27	32	·78	·10	·10	·01	—		
16	22	23	29	30	1	—	—	—	—	3·944	
18	19	25	26	32	·56	·38	·04	·02	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=4$ (*cont.*)						
17	19	26	28	30	·75	·25	—	—	—	3·974	
18	20	22	29	31	·59	·28	·13	—	—		
17	19	26	29	29	·75	·25	—	—	—	4·135	
17	19	27	27	30	·75	·25	—	—	—		
18	21	21	29	31	·60	·20	·20	—	—		
19	19	22	29	31	·43	·43	·14	—	—		
18	18	26	28	30	·50	·50	—	—	—	4·201	
18	20	22	30	30	·59	·28	·13	—	—		
16	21	25	28	30	1	—	—	—	—	4·211	
16	22	24	27	31	1	—	—	—	—		
17	21	24	26	32	·78	·14	·05	·02	—		
18	20	23	27	32	·60	·28	·10	·02	—		
17	20	24	28	31	·77	·19	·03	·01	—	4·284	
18	18	26	29	29	·50	·50	—	—	—	4·362	
18	18	27	27	30	·50	·50	—	—	—		
18	21	21	30	30	·60	·20	·20	—	—		
19	19	22	30	30	·43	·43	·14	—	—		
16	21	25	29	29	1	—	—	—	—	4·368	
16	23	23	27	31	1	—	—	—	—		
17	21	25	25	32	·78	·15	·04	·04	—		
19	19	23	27	32	·43	·43	·11	·02	—		
17	19	27	28	29	·75	·25	—	—	—	4·442	
19	20	21	29	31	·45	·32	·23	—	—		
16	21	26	27	30	1	—	—	—	—	4·502	
16	22	25	26	31	1	—	—	—	—		
17	22	23	26	32	·78	·11	·08	·03	—		
18	21	22	27	32	·61	·22	·16	·02	—		
17	19	28	28	28	·75	·25	—	—	—	4·589	
20	20	20	29	31	·33	·33	·33	—	—		
17	20	24	29	30	·77	·19	·03	—	—	4·664	
18	19	24	28	31	·56	·37	·05	·01	—		
18	18	27	28	29	·50	·50	—	—	—	4·670	
19	20	21	30	30	·45	·32	·23	—	—		
16	22	24	28	30	1	—	—	—	—	4·718	
18	20	24	26	32	·60	·29	·08	·04	—		
16	21	26	28	29	1	—	—	—	—	4·794	
16	23	24	26	31	1	—	—	—	—		
17	22	24	25	32	·78	·11	·06	·04	—		
19	20	22	27	32	·46	·33	·18	·03	—		
17	21	23	28	31	·78	·14	·07	·01	—	4·794	
17	20	25	27	31	·77	·19	·03	·01	—		
18	18	28	28	28	·50	·50	—	—	—	4·816	
20	20	20	30	30	·33	·33	·33	—	—		
16	22	24	29	29	1	—	—	—	—	4·872	
16	23	23	28	30	1	—	—	—	—		
18	20	25	25	32	·60	·29	·06	·06	—		
19	19	24	26	32	·44	·44	·09	·04	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=4$ (*cont.*)						
16	21	27	27	29	1	—	—	—	—	4·928	
16	23	25	25	31	1	—	—	—	—		
17	23	23	25	32	·78	·08	·08	·05	—		
19	21	21	27	32	·47	·25	·25	·03	—		
17	22	22	28	31	·78	·10	·10	·01	—	4·949	
17	20	26	26	31	·77	·19	·02	·02	—		
16	23	23	29	29	1	—	—	—	—	5·026	
19	19	25	25	32	·44	·44	·06	·06	—		
18	19	24	29	30	·56	·37	·05	·01	—	5·045	
16	21	27	28	28	1	—	—	—	—	5·063	
16	24	24	25	31	1	—	—	—	—		
17	23	24	24	32	·79	·09	·06	·06	—		
20	20	21	27	32	·35	·35	·26	·03	—		
16	22	25	27	30	1	—	—	—	—	5·130	
18	21	23	26	32	·61	·22	·12	·04	—		
18	19	25	27	31	·56	·37	·04	·02	—	5·173	
17	21	23	29	30	·78	·14	·07	·01	—		
16	22	26	26	30	1	—	—	—	—	5·262	
18	22	22	26	32	·62	·17	·17	·05	—		
18	20	23	28	31	·60	·29	·10	·01	—	5·307	
17	20	25	28	30	·77	·19	·03	·01	—		
17	22	22	29	30	·78	·10	·10	·01	—	5·328	
18	19	26	26	31	·57	·38	·03	·03	—		
16	22	25	28	29	1	—	—	—	—	5·414	
16	23	24	27	30	1	—	—	—	—		
18	21	24	25	32	·61	·23	·09	·07	—		
19	20	23	26	32	·47	·34	·14	·05	—		
17	21	24	27	31	·78	·15	·05	·02	—	5·424	
19	19	23	28	31	·43	·43	·11	·02	—	5·464	
17	20	25	29	29	·77	·19	·03	·01	·01		
18	21	22	28	31	·61	·22	·16	·02	—	5·596	
17	20	26	27	30	·77	·19	·02	·01	·01		
16	22	26	27	29	1	—	—	—	—	5·668	
16	23	25	26	30	1	—	—	—	—		
18	22	23	25	32	·62	·18	·13	·07	—		
19	21	22	26	32	·48	·26	·20	·06	—		
18	19	25	28	30	·57	·37	·04	·01	·01	5·686	
18	20	23	29	30	·60	·28	·10	·01	·01		
16	23	24	28	29	1	—	—	—	—	5·694	
19	20	24	25	32	·47	·34	·11	·08	—		
17	22	23	27	31	·78	·11	·08	·02	—	5·708	
17	21	25	26	31	·78	·15	·04	·03	—		
16	22	26	28	28	1	—	—	—	—	5·797	
16	24	24	26	30	1	—	—	—	—		
18	22	24	24	32	·62	·18	·10	·10	—		
20	20	22	26	32	·37	·37	·21	·06	—		
18	19	25	29	29	·57	·37	·04	·01	·01	5·843	
19	19	23	29	30	·43	·43	·11	·01	·01		
17	20	26	28	29	·77	·19	·02	·01	·01	5·887	
19	20	22	28	31	·46	·33	·18	·02	—		
16	22	27	27	28	1	—	—	—	—	5·920	
16	24	25	25	30	1	—	—	—	—		
18	23	23	24	32	·62	·14	·14	·10	—		
20	21	21	26	32	·37	·28	·28	·06	—		
17	21	24	28	30	·78	·15	·05	·01	·01	5·930	
18	20	24	27	31	·60	·29	·08	·03	·01		
18	19	26	27	30	·57	·38	·03	·02	·01	5·975	
18	21	22	29	30	·61	·22	·16	·01	·01		
19	21	21	28	31	·47	·25	·25	·02	·01	6·022	
17	20	27	27	29	·77	·19	·02	·02	·01		
16	23	25	27	29	1	—	—	—	—	6·064	
19	21	23	25	32	·49	·27	·16	·09	—		
17	21	24	29	29	·78	·15	·05	·01	·01	6·084	
19	19	24	27	31	·44	·44	·09	·03	·01		
17	22	24	26	31	·78	·11	·06	·03	—	6·114	
20	20	21	28	31	·35	·35	·26	·02	·01	6·156	
17	20	27	28	28	·77	·19	·02	·01	·01		
16	23	26	26	29	1	—	—	—	—	6·183	
19	22	22	25	32	·49	·21	·21	·09	—		
16	23	25	28	28	1	—	—	—	—	6·190	
16	24	24	27	28	1	—	—	—	—		
19	21	24	24	32	·49	·27	·12	·12	—		
20	20	23	25	32	·37	·37	·16	·09	—		
17	22	23	28	30	·78	·11	·08	·02	·01	6·210	
18	20	25	26	31	·60	·29	·06	·04	·01		
17	23	23	26	31	·78	·09	·09	·04	·01	6·243	
17	22	25	25	31	·78	·12	·05	·05	·01		
19	20	22	29	30	·46	·33	·18	·01	·01	6·266	
18	19	26	28	29	·57	·38	·03	·02	·01		
16	24	24	28	28	1	—	—	—	—	6·316	
20	20	24	24	32	·38	·38	·12	·12	—		
17	21	25	27	30	·78	·15	·04	·02	·01	6·334	
18	21	23	27	31	·61	·22	·12	·03	·01		
17	22	23	29	29	·78	·11	·08	·01	·01	6·364	
19	19	25	26	31	·44	·44	·07	·05	·01		
18	19	27	27	29	·57	·38	·02	·02	·01	6·399	·0000
19	21	21	29	30	·47	·25	·25	·02	·01		
16	23	26	27	28	1	—	—	—	—	6·426	·0001
16	24	25	26	29	1	—	—	—	—		
19	22	23	24	32	·50	·21	·16	·12	—		
20	21	22	25	32	·39	·29	·22	·10	—		
18	20	24	28	30	·60	·29	·08	·02	·01	6·435	·0001
17	21	26	26	30	·78	·15	·03	·03	·01	6·463	·0001
18	22	22	27	31	·62	·17	·17	·04	·01		
17	23	24	25	31	·79	·09	·07	·05	·01	6·494	·0001
18	19	27	28	28	·57	·38	·02	·02	·02	6·534	·0001
20	20	21	29	30	·36	·36	·26	·02	·01		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=4$ (*cont.*)						
16	23	27	27	27	1	—	—	—	—	6·543	·0001
16	25	25	25	29	1	—	—	—	—		
19	23	23	23	32	·50	·17	·17	·17	—		
21	21	21	25	32	·30	·30	·30	·10	—		
18	20	24	29	29	·59	·29	·08	·02	·02	6·589	·0001
19	19	24	28	30	·44	·44	·09	·02	·01		
17	24	24	24	31	·78	·07	·07	·07	·01	6·616	·0001
17	21	25	28	29	·78	·15	·04	·02	·01	6·617	·0001
19	20	23	27	31	·47	·34	·14	·04	·01		
16	24	25	27	28	1	—	—	—	—	6·665	·0001
20	21	23	24	32	·39	·30	·18	·13	—		
17	22	24	27	30	·78	·11	·06	·03	·01	6·730	·0001
18	21	24	26	31	·61	·23	·10	·05	·01		
19	19	24	29	29	·44	·44	·09	·02	·02	6·743	·0001
16	24	26	26	28	1	—	—	—	—	6·778	·0001
20	22	22	24	32	·40	·23	·23	·14	—		
18	20	25	27	30	·60	·29	·06	·03	·01	6·836	·0001
18	21	23	28	30	·61	·22	·12	·03	·01		
17	23	23	27	30	·78	·09	·09	·03	·01	6·856	·0001
18	21	25	25	31	·61	·23	·07	·07	·01		
17	21	26	27	29	·78	·15	·03	·02	·01	6·867	·0002
19	21	22	27	31	·48	·27	·20	·04	·01		
16	24	26	27	27	1	—	—	—	—	6·892	·0002
16	25	25	26	28	1	—	—	—	—		
20	22	23	23	32	·40	·24	·18	·18	—		
21	21	22	24	32	·31	·31	·24	·14	—		
18	20	26	26	30	·60	·29	·05	·05	·01	6·963	·0002
18	22	22	28	30	·62	·17	·17	·03	·01		
17	22	25	26	30	·78	·12	·05	·04	·01	6·975	·0002
18	22	23	26	31	·62	·18	·14	·06	·01		
19	19	25	27	30	·44	·44	·07	·04	·01	6·988	·0002
18	21	23	29	29	·61	·22	·12	·02	·02		
20	20	22	27	31	·37	·37	·21	·05	·01	6·995	·0002
17	21	26	28	28	·78	·15	·03	·02	·02		
16	25	25	27	27	1	—	—	—	—	7·005	·0002
21	21	23	23	32	·31	·31	·19	·19	—		
17	22	24	28	29	·78	·12	·06	·02	·02	7·008	·0003
19	20	24	26	31	·47	·35	·11	·06	·01		
16	25	26	26	27	1	—	—	—	—	7·115	·0003
18	22	22	29	29	·62	·17	·17	·02	·02		
19	19	26	26	30	·44	·44	·05	·05	·01		
21	22	22	23	32	·32	·25	·25	·19	—		
20	21	21	27	31	·38	·28	·28	·05	·01	7·117	·0003
17	21	27	27	28	·78	·15	·03	·03	·02		
18	20	25	28	29	·60	·29	·06	·03	·02	7·117	·0003
19	20	23	28	30	·47	·34	·14	·03	·02		
19	20	25	25	31	·47	·35	·09	·09	·01	7·133	·0003
18	23	23	28	29	·78	·09	·09	·02	·02		
18	22	24	25	31	·62	·18	·11	·08	·01	7·216	·0004
17	23	24	26	30	·79	·09	·07	·04	·01		
16	26	26	26	26	1	—	—	—	—	7·225	·0004
22	22	22	22	32	·25	·25	·25	·25	—		
19	20	23	29	29	·47	·34	·14	·02	·02	7·270	·0004
19	19	25	28	29	·44	·44	·07	·03	·02		
17	23	25	25	30	·79	·09	·05	·05	·01	7·332	·0004
18	23	23	25	31	·62	·14	·14	·08	·01		
18	21	24	27	30	·61	·23	·10	·04	·02	7·344	·0004
19	21	22	28	30	·48	·27	·20	·03	·02	7·365	·0005
18	20	26	27	29	·60	·29	·05	·04	·02		
19	21	23	26	31	·49	·27	·16	·07	·01	7·367	·0005
17	22	25	27	29	·78	·12	·05	·03	·02		
17	24	24	25	30	·79	·07	·07	·06	·01	7·447	·0005
18	23	24	24	31	·63	·14	·11	·11	·01		
19	22	22	26	31	·49	·21	·21	·07	·01	7·483	·0005
17	22	26	26	29	·78	·12	·04	·04	·02		
17	22	25	28	28	·78	·12	·05	·02	·02	7·492	·0006
20	20	23	26	31	·37	·37	·17	·07	·01		
18	20	26	28	28	·60	·29	·05	·03	·03	7·494	·0006
20	20	22	28	30	·37	·37	·21	·04	·02		
19	21	22	29	29	·48	·27	·20	·03	·03	7·518	·0006
19	19	26	27	29	·44	·44	·06	·04	·02		
18	21	25	26	30	·62	·23	·08	·06	·02	7·583	·0007
18	22	23	27	30	·62	·18	·14	·05	·02		
17	23	24	27	29	·79	·09	·07	·03	·02	7·604	·0007
19	21	24	25	31	·49	·28	·13	·10	·01		
18	20	27	27	28	·60	·29	·04	·04	·03	7·614	·0008
20	21	21	28	30	·38	·28	·28	·04	·02		
18	21	24	28	29	·61	·23	·10	·03	·02	7·621	·0008
19	20	24	27	30	·47	·35	·11	·05	·02		
19	19	26	28	28	·44	·44	·06	·03	·03	7·645	·0008
20	20	22	29	29	·37	·37	·21	·03	·03		
20	21	22	26	31	·39	·30	·23	·08	·01	7·723	·0009
17	22	26	27	28	·79	·12	·04	·03	·02		
20	20	24	25	31	·38	·38	·13	·10	·01	7·727	·0010
17	23	24	28	28	·79	·09	·07	·02	·02		
20	21	21	29	29	·38	·28	·28	·03	·03	7·766	·0010
19	19	27	27	28	·44	·44	·04	·04	·03		
17	23	25	26	29	·79	·09	·06	·04	·02	7·829	·0011
19	22	23	25	31	·50	·22	·17	·10	·01		
21	21	21	26	31	·30	·30	·30	·08	·01	7·838	·0011
17	22	27	27	27	·78	·12	·03	·03	·03		
18	22	23	28	29	·62	·18	·14	·04	·03	7·858	·0012
19	20	25	26	30	·47	·35	·09	·07	·02		
19	20	24	28	29	·47	·35	·11	·04	·03	7·897	·0012
18	22	24	26	30	·62	·18	·11	·06	·02	7·931	·0013
17	24	24	26	29	·79	·07	·07	·04	·02	7·940	·0014
19	22	24	24	31	·50	·22	·13	·13	·01		
18	21	25	27	29	·62	·23	·08	·05	·03	7·974	·0015
19	21	23	27	30	·49	·27	·16	·06	·02		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=4$ (*cont.*)						
18	22	25	25	30	·63	·18	·09	·09	·02	8·043	·0016
18	23	23	26	30	·63	·14	·14	·07	·02		
19	23	23	24	31	·50	·17	·17	·14	·01	8·049	·0016
17	24	25	25	29	·79	·07	·06	·06	·02		
17	23	25	27	28	·79	·10	·06	·03	·03	8·063	·0018
20	21	23	25	31	·39	·30	·18	·11	·01		
18	21	26	26	29	·62	·23	·06	·06	·03	8·088	·0019
19	22	22	27	30	·49	·21	·21	·06	·02		
18	21	25	28	28	·62	·23	·08	·04	·04	8·098	·0020
20	20	23	27	30	·38	·38	·17	·06	·02		
17	23	26	26	28	·78	·10	·05	·05	·03	8·172	·0021
20	22	22	25	31	·40	·24	·24	·11	·01		
20	21	24	24	31	·40	·30	·14	·14	·01	8·174	·0021
17	24	24	27	28	·79	·08	·08	·04	·03		
19	20	25	27	29	·47	·35	·09	·05	·03	8·248	·0024
19	21	23	28	29	·49	·27	·16	·04	·03		
18	23	24	25	30	·63	·15	·11	·09	·02	8·264	·0025
17	23	26	27	27	·79	·10	·05	·04	·04	8·283	·0026
21	21	22	25	31	·31	·31	·24	·12	·01		
19	21	24	26	30	·49	·28	·13	·08	·02	8·315	·0028
18	22	24	27	29	·62	·18	·11	·05	·03		
18	21	26	27	28	·62	·23	·06	·05	·04	8·326	·0031
20	21	22	27	30	·39	·30	·23	·06	·02		
19	22	22	28	29	·49	·21	·21	·04	·03	8·361	·0033
19	20	26	26	29	·48	·35	·07	·07	·03		
18	24	24	24	30	·63	·12	·12	·12	·02	8·371	·0033
20	20	23	28	29	·38	·38	·17	·05	·03	8·372	·0034
19	20	25	28	28	·47	·35	·09	·04	·04		
20	22	23	24	31	·40	·24	·19	·15	·01	8·388	·0037
17	24	25	26	28	·79	·08	·06	·05	·03		
18	23	23	27	29	·63	·15	·15	·05	·03	8·426	·0039
19	21	25	25	30	·49	·28	·10	·10	·02		
18	22	24	28	28	·62	·19	·11	·04	·04	8·438	·0040
20	20	24	26	30	·38	·38	·14	·08	·02		
18	21	27	27	27	·62	·23	·05	·05	·05	8·439	·0041
21	21	21	27	30	·30	·30	·30	·06	·02		
17	25	25	25	28	·79	·06	·06	·06	·03	8·494	·0041
20	23	23	23	31	·41	·19	·19	·19	·01		
17	24	25	27	27	·79	·08	·06	·04	·04	8·498	·0043
21	21	23	24	31	·32	·32	·20	·15	·02		
18	22	25	26	29	·63	·19	·09	·07	·03	8·534	·0047
19	22	23	26	30	·50	·22	·17	·08	·02		
20	20	25	25	30	·38	·38	·11	·11	·02	8·548	·0048
18	23	23	28	28	·63	·15	·15	·04	·04		
20	21	22	28	29	·39	·30	·23	·05	·04	8·598	·0052
19	20	26	27	28	·48	·35	·07	·06	·04		
17	24	26	26	27	·79	·08	·05	·05	·04	8·603	·0055
21	22	22	24	31	·32	·25	·25	·16	·02		
19	21	24	27	29	·49	·28	·13	·06	·03	8·698	·0057
21	22	23	23	31	·33	·26	·20	·20	·02		
17	25	25	26	27	·79	·06	·06	·05	·04	8·707	·0060
21	21	21	28	29	·30	·30	·30	·05	·04		
19	20	27	27	27	·48	·35	·06	·06	·06	8·710	·0061
18	23	24	26	29	·63	·15	·12	·07	·03		
19	22	24	25	30	·50	·22	·14	·11	·03	8·748	·0067
18	22	25	27	28	·63	·19	·09	·05	·04		
20	21	23	26	30	·40	·30	·19	·09	·03	8·766	·0073
17	25	26	26	26	·79	·06	·05	·05	·05		
22	22	22	23	31	·26	·26	·26	·20	·02	8·811	·0074
19	21	24	28	28	·49	·28	·13	·05	·05		
20	20	24	27	29	·38	·38	·14	·06	·04	8·819	·0078
19	23	23	25	30	·51	·18	·18	·11	·03		
18	23	25	25	29	·63	·15	·09	·09	·03	8·853	·0081
18	22	26	26	28	·63	·19	·07	·07	·04		
20	22	22	26	30	·40	·24	·24	·09	·03	8·873	·0085
19	21	25	26	29	·49	·28	·11	·08	·04		
19	22	23	27	29	·50	·22	·17	·06	·04	8·913	·0093
20	20	24	28	28	·38	·38	·14	·05	·05	8·941	·0094
18	24	24	25	29	·63	·12	·12	·09	·03		
19	23	24	24	30	·51	·18	·14	·14	·03	8·957	·0099
18	23	24	27	28	·63	·15	·12	·06	·04		
20	21	24	25	30	·40	·31	·15	·12	·03	8·978	·0108
18	22	26	27	27	·63	·19	·07	·06	·06		
21	21	22	26	30	·32	·32	·25	·09	·03	8·982	·0113
20	20	25	26	29	·38	·38	·11	·09	·04		
19	22	23	28	28	·50	·22	·17	·05	·05	9·034	·0118
20	21	23	27	29	·40	·31	·19	·07	·04		
19	21	25	27	28	·50	·28	·11	·07	·05	9·144	·0130
20	22	23	25	30	·41	·25	·20	·12	·03		
18	23	25	26	28	·63	·15	·10	·08	·05	9·185	·0145
19	22	24	26	29	·50	·23	·14	·09	·04	9·228	·0152
19	21	26	26	28	·50	·28	·08	·08	·05		
20	22	22	27	29	·40	·24	·24	·07	·04	9·249	·0160
20	20	25	27	28	·38	·38	·11	·07	·05		
20	21	23	28	28	·40	·31	·19	·05	·05	9·264	·0168
18	24	24	26	28	·63	·12	·12	·08	·05		
20	22	24	24	30	·41	·25	·16	·16	·03	9·287	·0177
18	23	25	27	27	·63	·15	·10	·06	·06		
21	21	23	25	30	·32	·32	·20	·13	·03	9·293	·0186
19	22	25	25	29	·51	·23	·11	·11	·04		
19	23	23	26	29	·51	·18	·18	·09	·04	9·331	·0195
19	21	26	27	27	·50	·28	·09	·07	·07		
21	21	22	27	29	·32	·32	·25	·07	·04	9·358	·0205
20	20	26	26	28	·38	·38	·09	·09	·05		
20	22	22	28	28	·40	·24	·24	·06	·06	9·369	·0210
18	24	25	25	28	·63	·12	·10	·10	·05		
20	23	23	24	30	·41	·20	·20	·16	·03	9·388	·0221
21	21	24	24	30	·32	·32	·16	·16	·03		
18	24	24	27	27	·63	·12	·12	·06	·06	9·394	·0226

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=4$ (*cont.*)											
18	23	26	26	27	·63	·15	·08	·08	·06	9·395	·0237
21	22	22	25	30	·33	·26	·26	·13	·03		
19	22	24	27	28	·51	·23	·14	·07	·05	9·456	·0261
20	21	24	26	29	·40	·31	·15	·10	·04		
21	20	26	27	27	·38	·38	·09	·07	·07	9·478	·0267
21	21	22	28	28	·32	·32	·25	·06	·06		
19	23	24	25	29	·51	·18	·15	·12	·04	9·533	·0282
19	23	23	27	28	·51	·18	·18	·07	·06	9·557	·0296
20	21	25	25	29	·40	·31	·12	·12	·04		
18	24	25	26	27	·64	·12	·10	·08	·06	9·595	·0329
21	22	23	24	30	·33	·26	·21	·17	·03		
19	24	24	24	29	·51	·15	·15	·15	·04	9·632	·0332
20	22	23	26	29	·41	·25	·20	·10	·04	9·659	·0369
19	22	25	26	28	·51	·23	·12	·09	·06		
20	21	24	27	28	·40	·31	·15	·08	·06	9·682	·0388
21	23	23	23	30	·33	·21	·21	·21	·03	9·694	·0394
18	25	25	25	27	·64	·10	·10	·10	·06		
18	24	26	26	26	·64	·13	·08	·08	·08	9·696	·0401
22	22	22	24	30	·27	·27	·27	·17	·03		
19	22	25	27	27	·51	·23	·12	·07	·07	9·765	·0424
21	21	23	26	29	·32	·32	·20	·10	·05		
22	22	23	23	30	·27	·27	·21	·21	·03	9·794	·0436
18	25	25	26	26	·64	·10	·10	·08	·08		
20	22	24	25	29	·41	·25	·16	·13	·05	9·858	·0491
19	23	24	26	28	·51	·19	·15	·09	·06		
19	22	26	26	27	·51	·23	·09	·09	·07	9·866	·0519
21	22	22	26	29	·33	·26	·26	·10	·05		
20	21	25	26	28	·40	·31	·12	·10	·06	9·883	·0576
20	22	23	27	28	·41	·25	·20	·08	·06		
20	23	23	25	29	·41	·20	·20	·13	·05	9·956	·0610
19	23	25	25	28	·51	·19	·12	·12	·06		
19	23	24	27	27	·51	·19	·15	·08	·08	9·963	·0644
21	21	24	25	29	·33	·33	·16	·13	·05		
21	21	23	27	28	·33	·33	·20	·08	·06	9·989	·0679
20	21	25	27	27	·40	·31	·13	·08	·08		
19	24	24	25	28	·52	·15	·15	·12	·06	10·053	·0720
20	23	24	24	29	·42	·21	·16	·16	·05		
20	21	26	26	27	·40	·31	·10	·10	·08	10·089	·0764
21	22	22	27	28	·33	·26	·26	·08	·06		
19	23	25	26	27	·51	·19	·12	·10	·08	10·159	·0866
21	22	23	25	29	·34	·27	·21	·14	·05		
20	22	24	26	28	·41	·26	·16	·10	·06	10·179	·0918
21	22	24	24	29	·34	·27	·17	·17	·05	10·255	·0980
19	24	24	26	27	·52	·15	·15	·10	·08		
19	23	26	26	26	·52	·19	·10	·10	·10	10·258	·1001
22	22	22	25	29	·27	·27	·27	·14	·05		
20	22	25	25	28	·41	·26	·13	·13	·07	10·276	·1065
20	23	23	26	28	·42	·21	·21	·11	·07		
20	22	24	27	27	·41	·26	·16	·08	·08	10·284	·1130
21	21	24	26	28	·33	·33	·17	·11	·07		
19	24	25	25	27	·52	·15	·12	·12	·08	10·351	·1206
21	23	23	24	29	·34	·22	·22	·17	·05		
20	23	23	27	27	·42	·21	·21	·08	·08	10·380	·1245
21	21	25	25	28	·33	·33	·14	·14	·07		
21	21	24	27	27	·33	·33	·17	·09	·09	10·387	·1265
19	24	25	26	26	·52	·16	·13	·10	·10	10·448	·1357
22	22	23	24	29	·27	·27	·22	·18	·05		
20	23	24	25	28	·42	·21	·17	·14	·07	10·467	·1452
20	22	25	26	27	·42	·26	·13	·11	·09	10·478	·1646
21	22	23	26	28	·34	·27	·21	·11	·07		
19	25	25	25	26	·52	·13	·13	·13	·10	10·542	·1684
22	23	23	23	29	·28	·22	·22	·22	·05		
20	24	24	24	28	·42	·17	·17	·17	·07	10·561	·1703
20	22	26	26	26	·42	·26	·11	·11	·11	10·575	·1742
22	22	22	26	28	·27	·27	·27	·11	·07		
21	22	23	27	27	·34	·27	·22	·09	·09	10·581	·1862
21	21	25	26	27	·33	·33	·14	·11	·09		
21	22	24	25	28	·34	·27	·18	·14	·07	10·666	·2148
20	23	24	26	27	·42	·21	·17	·11	·09		
21	21	26	26	26	·33	·33	·11	·11	·11	10·678	·2173
22	22	22	27	27	·27	·27	·27	·09	·09		
20	23	25	25	27	·42	·21	·14	·14	·09	10·760	·2346
21	23	23	25	28	·34	·22	·22	·14	·07		
20	24	24	25	27	·42	·17	·17	·14	·09	10·853	·2556
21	23	24	24	28	·35	·22	·18	·18	·07		
20	23	25	26	26	·42	·21	·14	·11	·11	10·856	·2768
22	22	23	25	28	·28	·28	·22	·15	·07		
21	22	24	26	27	·34	·27	·18	·12	·09	10·865	·2983
20	24	24	26	26	·43	·17	·17	·11	·11	10·948	·3111
22	22	24	24	28	·28	·28	·18	·18	·07		
21	22	25	25	27	·34	·27	·14	·14	·09	10·958	·3371
21	23	23	26	27	·35	·22	·22	·12	·09		
22	23	23	24	28	·28	·23	·23	·18	·08	11·040	·3680
20	24	25	25	26	·43	·17	·14	·14	·11		
21	22	25	26	26	·34	·28	·15	·12	·12	11·053	·3997
22	22	23	26	27	·28	·28	·23	·12	·10		
20	25	25	25	25	·43	·14	·14	·14	·14	11·132	·4028
23	23	23	23	28	·23	·23	·23	·23	·08		
21	23	24	25	27	·35	·23	·18	·15	·10	11·141	·4409
21	24	24	24	27	·35	·18	·18	·18	·10	11·232	·4485
22	22	24	25	27	·28	·28	·19	·15	·10	11·236	·4948
21	23	24	26	26	·35	·23	·18	·12	·12		
22	23	23	25	27	·29	·23	·23	·15	·10	11·327	·5506
21	23	25	25	26	·35	·23	·15	·15	·12		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=4$ (*cont.*)						
22	22	24	26	26	·28	·28	·19	·12	·12	11·330	·5646
22	23	24	24	27	·29	·23	·19	·19	·10	11·417	·6318
21	24	24	25	26	·35	·19	·19	·15	·12		
22	23	23	26	26	·29	·23	·23	·12	·12	11·421	·6657
22	22	25	25	26	·29	·29	·15	·15	·12		
23	23	23	24	27	·24	·24	·24	·19	·10	11·507	·6927
21	24	25	25	25	·35	·19	·15	·15	·15		
22	23	24	25	26	·29	·24	·19	·16	·13	11·600	·7909
22	24	24	24	26	·29	·19	·19	·19	·13	11·689	·8106
22	23	25	25	25	·29	·24	·16	·16	·16	11·690	·8500
23	23	23	25	26	·24	·24	·24	·16	·13		
22	24	24	25	25	·29	·19	·19	·16	·16	11·778	·9211
23	23	24	24	26	·24	·24	·20	·20	·13		
23	23	24	25	25	·24	·24	·20	·16	·16	11·866	·9638
23	24	24	24	25	·24	·20	·20	·20	·16	11·954	·9979
24	24	24	24	24	·20	·20	·20	·20	·20	12·041	1·0000
					$n=5$						
20	25	30	35	40	1	—	—	—	—	0	
20	25	30	36	39	1	—	—	—	—	1·087	
20	25	31	34	40	1	—	—	—	—		
20	26	29	35	40	1	—	—	—	—		
21	24	30	35	40	·80	·20	—	—	—		
20	25	30	37	38	1	—	—	—	—	1·461	
20	25	32	33	40	1	—	—	—	—		
20	27	28	35	40	1	—	—	—	—		
22	23	30	35	40	·60	·40	—	—	—		
20	26	29	36	39	1	—	—	—	—	2·173	
21	24	30	36	39	·80	·20	—	—	—		
21	24	31	34	40	·80	·20	—	—	—		
20	25	31	35	39	1	—	—	—	—	2·274	
20	26	30	34	40	1	—	—	—	—		
21	25	29	35	40	·81	·16	·03	—	—		
20	26	29	37	38	1	—	—	—	—	2·548	
20	27	28	36	39	1	—	—	—	—		
21	24	30	37	38	·80	·20	—	—	—		
21	24	32	33	40	·80	·20	—	—	—		
22	23	30	36	39	·60	·40	—	—	—		
22	23	31	34	40	·60	·40	—	—	—		
20	25	31	36	38	1	—	—	—	—	2·767	
20	25	32	34	39	1	—	—	—	—		
20	26	31	33	40	1	—	—	—	—		
20	27	29	34	40	1	—	—	—	—		
21	26	28	35	40	·82	·12	·06	—	—		
22	24	29	35	40	·64	·31	·05	—	—		
20	25	31	37	37	1	—	—	—	—	2·917	
20	25	33	33	39	1	—	—	—	—		
20	26	32	32	40	1	—	—	—	—		
20	28	28	34	40	1	—	—	—	—		
21	27	27	35	40	·82	·09	·09	—	—		
23	23	29	35	40	·47	·47	·05	—	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
20	27	28	37	38	1	—	—	—	—	2·923	
22	23	30	37	38	·60	·40	—	—	—		
22	23	32	33	40	·60	·40	—	—	—		
21	24	31	35	39	·80	·20	—	—	—	3·361	
21	25	29	36	39	·81	·16	·03	—	—		
20	25	32	35	38	1	—	—	—	—	3·373	
20	27	30	33	40	1	—	—	—	—		
22	25	28	35	40	·66	·25	·09	—	—		
20	26	30	35	39	1	—	—	—	—	3·458	
21	25	30	34	40	·81	·16	·02	—	—		
20	25	32	36	37	1	—	—	—	—		
20	25	33	34	38	1	—	—	—	—	3·645	
20	27	31	32	40	1	—	—	—	—		
20	28	29	33	40	1	—	—	—	—		
22	26	27	35	40	·67	·19	·14	—	—		
23	24	28	35	40	·52	·38	·11	—	—		
21	25	29	37	38	·81	·16	·03	—	—	3·736	
22	23	31	35	39	·60	·40	—	—	—		
21	24	31	36	38	·80	·20	—	—	—	3·854	
21	24	32	34	39	·80	·20	—	—	—		
21	26	28	36	39	·82	·12	·06	—	—		
22	24	29	36	39	·64	·31	·05	—	—		
20	26	30	36	38	1	—	—	—	—	3·948	
20	27	29	35	39	1	—	—	—	—		
21	25	31	33	40	·81	·16	·02	·01	—		
22	24	30	34	40	·64	·32	·04	·01	—		
21	24	31	37	37	·80	·20	—	—	—	4·004	
21	24	33	33	39	·80	·20	—	—	—		
21	27	27	36	39	·82	·09	·09	—	—		
23	23	29	36	39	·47	·47	·05	—	—		
20	25	33	35	37	1	—	—	—	—	4·035	
20	28	30	32	40	1	—	—	—	—		
23	25	27	35	40	·53	·30	·17	—	—		
20	26	31	34	39	1	—	—	—	—	4·053	
21	26	29	34	40	·82	·13	·05	·01	—		
20	26	30	37	37	1	—	—	—	—	4·098	
20	28	28	35	39	1	—	—	—	—		
21	25	32	32	40	·81	·16	·01	·01	—		
23	23	30	34	40	·47	·47	·04	·01	—		
20	25	33	36	36	1	—	—	—	—	4·158	
20	25	34	34	37	1	—	—	—	—		
20	28	31	31	40	1	—	—	—	—		
20	29	29	32	40	1	—	—	—	—		
23	26	26	35	40	·54	·23	·23	—	—		
24	24	27	35	40	·41	·41	·18	—	—		
21	26	28	37	38	·82	·12	·06	—	—	4·228	
22	23	31	36	38	·60	·40	—	—	—		
22	23	32	34	39	·60	·40	—	—	—		
22	24	29	37	38	·64	·31	·05	—	—		
20	26	32	33	39	1	—	—	—	—	4·321	
21	27	28	34	40	·82	·10	·07	·01	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=5$ (*cont.*)											
21	27	27	37	38	·82	·09	·09	—	—	4·378	
22	23	31	37	37	·60	·40	—	—	—		
22	23	33	33	39	·60	·40	—	—	—		
23	23	29	37	38	·47	·47	·05	—	—		
20	25	34	35	36	1	—	—	—	—	4·398	
20	29	30	31	40	1	—	—	—	—		
24	25	26	35	40	·43	·32	·25	—	—		
20	27	29	36	38	1	—	—	—	—	4·437	
22	24	31	33	40	·64	·32	·03	·01	—		
21	24	32	35	38	·80	·20	—	—	—	4·459	
22	25	28	36	39	·66	·25	·09	—	—		
20	25	35	35	35	1	—	—	—	—	4·516	
20	30	30	30	40	1	—	—	—	—		
25	25	25	35	40	·33	·33	·33	—	—		
20	27	29	37	37	1	—	—	—	—	4·586	
20	28	28	36	38	1	—	—	—	—		
22	24	32	32	40	·64	·31	·02	·02	—		
23	23	31	33	40	·47	·47	·03	·02	—		
21	25	30	35	39	·81	·16	·02	—	—	4·642	
20	26	31	35	38	1	—	—	—	—	4·649	
20	27	30	34	39	1	—	—	—	—		
21	26	30	33	40	·82	·13	·04	·02	—		
22	25	29	34	40	·66	·26	·08	·01	—		
21	24	32	36	37	·80	·20	—	—	—	4·732	
21	24	33	34	38	·80	·20	—	—	—		
22	26	27	36	39	·67	·19	·14	—	—		
23	24	28	36	39	·52	·38	·11	—	—		
20	28	28	37	37	1	—	—	—	—	4·735	
23	23	32	32	40	·48	·48	·02	·02	—		
22	23	32	35	38	·60	·40	—	—	—	4·834	
22	25	28	37	38	·66	·25	·09	—	—		
20	26	31	36	37	1	—	—	—	—	4·917	
20	28	29	34	39	1	—	—	—	—		
21	26	31	32	40	·82	·13	·03	·02	—		
23	24	29	34	40	·52	·38	·09	·01	—		
20	26	32	34	38	1	—	—	—	—	5·024	
20	27	31	33	39	1	—	—	—	—		
21	27	29	33	40	·82	·10	·06	·02	—		
22	26	28	34	40	·67	·20	·12	·01	—		
22	23	32	36	37	·60	·40	—	—	—	5·107	
22	23	33	34	38	·60	·40	—	—	—		
22	26	27	37	38	·67	·19	·14	—	—		
23	24	28	37	38	·52	·38	·11	—	—		
21	24	33	35	37	·80	·20	—	—	—	5·121	
23	25	27	36	39	·53	·30	·17	—	—		
21	25	30	36	38	·81	·16	·02	—	—	5·132	
22	24	30	35	39	·64	·32	·04	·01	—		
20	26	33	33	38	1	—	—	—	—	5·144	
20	27	32	32	39	1	—	—	—	—		
21	28	28	33	40	·82	·08	·08	·02	—		
22	27	27	34	40	·67	·16	·16	·02	—		
21	25	31	34	39	·81	·16	·02	·01	—	5·234	
21	26	29	35	39	·82	·13	·05	·01	—		
20	27	30	35	38	1	—	—	—	—	5·239	
22	25	30	33	40	·66	·26	·06	·02	—		
21	24	33	36	36	·80	·20	—	—	—	5·245	
21	24	34	34	37	·80	·20	—	—	—		
23	26	26	36	39	·54	·23	·23	—	—		
24	24	27	36	39	·41	·41	·18	—	—		
21	25	30	37	37	·81	·16	·02	—	—	5·282	
23	23	30	35	39	·47	·47	·04	·01	—		
20	26	32	35	37	1	—	—	—	—	5·401	
20	28	30	33	39	1	—	—	—	—		
21	27	30	32	40	·82	·11	·05	·03	—		
23	25	28	34	40	·54	·31	·14	·02	—		
21	24	34	35	36	·80	·20	—	—	—	5·485	
24	25	26	36	39	·43	·32	·25	—	—		
22	23	33	35	37	·60	·40	—	—	—	5·496	
23	25	27	37	38	·53	·30	·17	—	—		
21	25	32	33	39	·81	·16	·02	·01	—	5·501	
21	27	28	35	39	·82	·10	·07	·01	—		
20	27	30	36	37	1	—	—	—	—	5·505	
20	28	29	35	38	1	—	—	—	—		
22	25	31	32	40	·66	·26	·05	·04	—		
23	24	30	33	40	·52	·38	·07	·03	—		
20	26	32	36	36	1	—	—	—	—	5·521	
20	29	29	33	39	1	—	—	—	—		
21	27	31	31	40	·82	·11	·04	·04	—		
24	24	28	34	40	·42	·42	·14	·02	—		
21	24	35	35	35	·80	·20	—	—	—	5·602	
25	25	25	36	39	·33	·33	·33	—	—		
22	23	33	36	36	·60	·40	—	—	—	5·620	
22	23	34	34	37	·60	·40	—	—	—		
23	26	26	37	38	·54	·23	·23	—	—		
24	24	27	37	38	·41	·41	·18	—	—		
22	24	30	36	38	·64	·32	·04	—	—	5·622	
20	26	33	34	37	1	—	—	—	—	5·629	
20	28	31	32	39	1	—	—	—	—		
21	28	29	32	40	·82	·08	·06	·03	—		
23	26	27	34	40	·55	·24	·19	·02	—		
20	27	31	34	38	1	—	—	—	—	5·714	
22	26	29	33	40	·67	·21	·09	·03	—		
21	26	29	36	38	·82	·13	·05	—	—	5·723	
22	24	31	34	39	·64	·32	·03	·01	—		
20	28	29	36	37	1	—	—	—	—	5·770	
23	24	31	32	40	·52	·38	·06	·04	—		
22	24	30	37	37	·64	·32	·04	—	—	5·771	
23	23	30	36	38	·47	·47	·04	·01	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=5$ (*cont.*)											
21	25	31	35	38	·81	·16	·02	·01	—	5·830	
22	25	29	35	39	·66	·25	·08	·01	—		
20	26	33	35	36	1	—	—	—	—	5·858	
20	29	30	32	39	1	—	—	—	—		
21	28	30	31	40	·82	·09	·05	·04	—		
24	25	27	34	40	·44	·34	·20	·02	—		
22	23	34	35	36	·60	·40	—	—	—	5·860	
24	25	26	37	38	·43	·32	·25	—	—		
21	26	29	37	37	·82	·13	·05	—	—	5·872	
23	23	31	34	39	·47	·47	·03	·01	—		
23	23	30	37	37	·47	·47	·04	—	—	5·920	
21	26	30	34	39	·82	·13	·04	·01	—	5·925	
20	27	32	33	38	1	—	—	—	—	5·938	
22	27	28	33	40	·68	·17	·13	·03	—		
20	26	34	34	36	1	—	—	—	—	5·966	
20	29	31	31	39	1	—	—	—	—		
21	29	29	31	40	·82	·07	·07	·04	—		
24	26	26	34	40	·44	·27	·27	·02	—		
22	23	35	35	35	·60	·40	—	—	—	5·977	
25	25	25	37	38	·33	·33	·33	—	—		
21	27	28	36	38	·82	·10	·07	·01	—	5·989	
22	24	32	33	39	·64	·32	·02	·02	—		
20	26	34	35	35	1	—	—	—	—	6·075	
20	30	30	31	39	1	—	—	—	—		
21	29	30	30	40	·82	·07	·05	·05	—		
25	25	26	34	40	·35	·35	·27	·02	—		
20	27	31	35	37	1	—	—	—	—	6·083	
20	28	30	34	38	1	—	—	—	—		
22	26	30	32	40	·67	·21	·08	·04	—		
23	25	29	33	40	·54	·31	·11	·04	—		
21	25	31	36	37	·81	·16	·02	—	—	6·098	
23	24	29	35	39	·52	·38	·09	·01	—		
21	27	28	37	37	·82	·10	·07	—	—	6·138	
23	23	32	33	39	·48	·48	·03	·02	—		
20	27	31	36	36	1	—	—	—	—	6·201	
20	29	29	34	38	1	—	—	—	—		
22	26	31	31	40	·67	·21	·06	·06	—		
24	24	29	33	40	·42	·42	·12	·04	—		
21	25	32	34	38	·81	·16	·02	·01	—	6·204	
22	26	28	35	39	·67	·20	·12	·01	—		
21	26	31	33	39	·82	·13	·03	·02	—	6·295	
21	27	29	34	39	·82	·10	·06	·01	—		
22	24	31	35	38	·64	·32	·03	·01	—	6·318	
22	25	29	36	38	·66	·25	·08	·01	—		
21	25	33	33	38	·81	·16	·01	·01	—	6·323	
22	27	27	35	39	·67	·16	·16	·01	—		
20	27	32	34	37	1	—	—	—	—	6·408	
20	28	31	33	38	1	—	—	—	—		
22	27	29	32	40	·68	·17	·10	·05	—		
23	26	28	33	40	·55	·25	·16	·04	—		
21	26	32	32	39	·82	·13	·02	·02	—	6·413	
21	28	28	34	39	·82	·08	·08	·02	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=5$ (*cont.*)											
22	25	29	37	37	·66	·25	·08	·01	·01	6·468	
23	23	31	35	38	·48	·48	·04	·01	—		
20	28	30	35	37	1	—	—	—	—	6·479	
23	25	30	32	40	·54	·32	·09	·05	—		
20	27	33	33	37	1	—	—	—	—	6·512	
20	28	32	32	38	1	—	—	—	—		
22	28	28	32	40	·68	·14	·14	·05	—		
23	27	27	33	40	·56	·20	·20	·04	—		
21	26	30	35	38	·82	·13	·04	·01	—	6·514	
22	25	30	34	39	·66	·26	·06	·02	—		
20	28	30	36	36	1	—	—	—	—	6·566	
20	29	29	35	37	1	—	—	—	—		
23	25	31	31	40	·54	·32	·07	·07	—		
24	24	30	32	40	·42	·42	·10	·06	—		
21	25	32	35	37	·81	·16	·02	·01	—	6·581	
23	25	28	35	39	·54	·31	·14	·01	—		
22	24	31	36	37	·64	·32	·03	·01	·01	6·587	
23	24	29	36	38	·52	·38	·09	·01	—		
20	27	32	35	36	1	—	—	—	—	6·629	
20	29	30	33	38	1	—	—	—	—		
22	27	30	31	40	·68	·17	·08	·07	—		
24	25	28	33	40	·44	·34	·17	·04	—		
20	29	29	36	36	1	—	—	—	—	6·682	
24	24	31	31	40	·42	·42	·08	·08	—		
22	24	32	34	38	·64	·32	·03	·01	—	6·692	
22	26	28	36	38	·67	·20	·12	·01	—		
21	25	32	36	36	·81	·16	·02	·01	·01	6·700	
24	24	28	35	39	·42	·42	·14	·02	—		
23	23	31	36	37	·48	·48	·04	·01	·01	6·736	
23	24	29	37	37	·52	·38	·09	·01	·01		
21	27	30	33	39	·82	·11	·05	·02	—	6·764	
21	26	30	36	37	·82	·13	·04	·01	·01	6·780	
23	24	30	34	39	·52	·38	·07	·02	—		
21	25	33	34	37	·81	·16	·01	·01	—	6·808	
23	26	27	35	39	·55	·24	·19	·02	—		
22	24	33	33	38	·64	·32	·02	·02	—	6·811	
22	27	27	36	38	·67	·16	·16	·01	—		
20	27	33	34	36	1	—	—	—	—	6·835	
20	29	31	32	38	1	—	—	—	—		
22	28	29	31	40	·68	·14	·11	·07	—		
24	26	27	33	40	·45	·28	·22	·05	—		
22	26	28	37	37	·67	·20	·12	·01	·01	6·840	
23	23	32	34	38	·48	·48	·03	·02	—		
20	28	31	34	37	1	—	—	—	—	6·869	
23	26	29	32	40	·55	·26	·13	·06	—		
21	27	29	35	38	·82	·10	·06	·01	—	6·882	
22	25	31	33	39	·66	·26	·05	·03	—		
20	27	33	35	35	1	—	—	—	—	6·939	
20	30	30	32	38	1	—	—	—	—		
22	28	30	30	40	·68	·14	·09	·09	—		
25	25	27	33	40	·36	·36	·23	·05	—		
22	27	27	37	37	·67	·16	·16	·01	·01	6·960	
23	23	33	33	38	·48	·48	·02	·02	·01		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
21	26	31	34	38	·82	·13	·03	·02	—	6·983	
22	26	29	34	39	·67	·21	·10	·02	—		
21	27	31	32	39	·82	·11	·04	·03	—	6·987	
21	28	29	33	39	·82	·08	·07	·02	—		
21	28	28	35	38	·82	·08	·08	·01	—	6·998	
22	25	32	32	39	·66	·26	·04	·04	—		
21	25	33	35	36	·81	·16	·01	·01	·01	7·036	
24	25	27	35	39	·44	·34	·20	·02	—		
20	27	34	34	35	1	—	—	—	—	7·040	
20	30	31	31	38	1	—	—	—	—		
22	29	29	30	40	·68	·11	·11	·09	—		
25	26	26	33	40	·37	·29	·29	·05	—		
22	24	32	35	37	·64	·32	·03	·01	·01	7·068	
23	25	28	36	38	·54	·31	·14	·01	·01		
20	28	32	33	37	1	—	—	—	—	7·071	
23	27	28	32	40	·56	·21	·17	·06	—		
20	28	31	35	36	1	—	—	—	—	7·087	
20	29	30	34	37	1	—	—	—	—		
23	26	30	31	40	·56	·26	·10	·08	—		
24	25	29	32	40	·45	·35	·14	·06	—		
22	25	30	35	38	·66	·26	·06	·02	·01	7·104	
21	25	34	34	36	·81	·16	·01	·01	·01	7·145	
24	26	26	35	39	·44	·27	·27	·02	—		
21	27	29	36	37	·82	·10	·06	·01	·01	7·146	
23	24	31	33	39	·52	·38	·06	·03	—		
22	24	32	36	36	·64	·32	·03	·01	·01	7·188	
24	24	28	36	38	·42	·42	·15	·01	·01		
21	26	32	33	38	·82	·13	·03	·02	—	7·205	
22	27	28	34	39	·68	·17	·13	·03	—		
23	23	32	35	37	·48	·48	·03	·01	·01	7·217	
23	25	28	37	37	·54	·31	·14	·01	·01		
21	25	34	35	35	·81	·16	·01	·01	·01	7·254	
25	25	26	35	39	·35	·35	·27	·02	—		
21	28	28	36	37	·82	·08	·08	·01	·01	7·263	
23	24	32	32	39	·52	·38	·05	·05	—		
22	24	33	34	37	·64	·32	·02	·02	·01	7·295	
23	26	27	36	38	·55	·24	·19	·01	·01		
20	29	30	35	36	1	—	—	—	—	7·304	
24	25	30	31	40	·45	·35	·11	·09	—		
21	28	30	32	39	·82	·09	·05	·03	—	7·308	
23	23	32	36	36	·48	·48	·03	·01	·01	7·337	
24	24	28	37	37	·42	·42	·15	·01	·01		
21	26	31	35	37	·82	·13	·03	·01	·01	7·352	
23	25	29	34	39	·54	·31	·11	·03	—		
22	25	30	36	37	·66	·26	·06	·01	·01	7·370	
23	24	30	35	38	·52	·38	·07	·02	·01		
20	28	32	34	36	1	—	—	—	—	7·385	
20	29	31	33	37	1	—	—	—	—		
23	27	29	31	40	·56	·21	·14	·09	—		
24	26	28	32	40	·46	·29	·18	·07	—		
21	28	31	31	39	·82	·09	·04	·04	—	7·413	
21	29	29	32	39	·82	·07	·07	·03	—		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
23	23	33	34	37	·48	·48	·02	·02	·01	7·444	
23	26	27	37	37	·55	·25	·19	·01	·01		
21	27	30	34	38	·82	·11	·05	·02	·01	7·445	
22	26	30	33	39	·67	·21	·08	·04	—		
21	26	31	36	36	·82	·13	·03	·01	·01	7·470	
24	24	29	34	39	·42	·42	·12	·03	—		
20	28	33	33	36	1	—	—	—	—	7·482	
20	29	32	32	37	1	—	—	—	—		
23	28	28	31	40	·57	·17	·17	·09	—		
24	27	27	32	40	·46	·23	·23	·07	—		
20	28	32	35	35	1	—	—	—	—	7·486	
20	30	30	33	37	1	—	—	—	—		
23	27	30	30	40	·57	·21	·11	·11	—		
25	25	28	32	40	·37	·37	·19	·07	—		
22	24	33	35	36	·64	·32	·02	·01	·01	7·524	
24	25	27	36	38	·44	·34	·20	·01	·01		
22	25	31	34	38	·66	·26	·05	·02	·01	7·569	
22	26	29	35	38	·67	·21	·10	·02	·01		
21	29	30	31	39	·82	·07	·06	·04	—	7·617	
22	24	34	34	36	·64	·32	·02	·02	·01	7·632	
24	26	26	36	38	·44	·27	·27	·01	·01		
23	24	30	36	37	·52	·38	·07	·01	·01	7·636	
21	28	29	34	38	·82	·09	·07	·02	·01	7·663	
22	26	31	32	39	·67	·21	·06	·05	·01		
23	23	33	35	36	·48	·48	·02	·01	·01	7·672	
24	25	27	37	37	·44	·34	·20	·01	·01		
21	26	32	34	37	·82	·13	·03	·02	·01	7·673	
23	26	28	34	39	·55	·25	·16	·03	·01		
20	28	33	34	35	1	—	—	—	—	7·679	
20	30	31	32	37	1	—	—	—	—		
23	28	29	30	40	·57	·18	·14	·11	—		
25	26	27	32	40	·38	·30	·24	·08	—		
20	29	31	34	36	1	—	—	—	—	7·694	
24	26	29	31	40	·46	·29	·15	·09	—		
21	30	30	30	39	·82	·06	·06	·06	—	7·717	
22	24	34	35	35	·64	·32	·02	·01	·01	7·741	
25	25	26	36	38	·35	·35	·27	·01	·01		
21	27	31	33	38	·82	·11	·04	·03	·01	7·761	
22	27	29	33	39	·68	·17	·11	·04	·01		
20	28	34	34	34	1	—	—	—	—	7·775	
20	31	31	31	37	1	—	—	—	—		
23	29	29	29	40	·57	·14	·14	·14	—		
26	26	26	32	40	·31	·31	·31	·08	—		
21	26	33	33	37	·82	·13	·02	·02	·01	7·777	
23	27	27	34	39	·56	·20	·20	·03	·01		
23	23	34	34	36	·48	·48	·02	·02	·01	7·780	
24	26	26	37	37	·44	·27	·27	·01	·01		
22	25	32	33	38	·66	·26	·04	·03	·01	7·789	
22	27	28	35	38	·68	·17	·13	·02	·01		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=5$ (*cont.*)											
20	29	31	35	35	1	—	—	—	—	7·795	
20	30	30	34	36	1	—	—	—	—		
24	26	30	30	40	·47	·29	·12	·12	—		
25	25	29	31	40	·37	·37	·16	·10	—		
21	27	30	35	37	·82	·11	·05	·01	·01	7·810	
23	25	30	33	39	·54	·32	·09	·04	·01		
22	26	29	36	37	·67	·21	·10	·02	·01	7·833	
23	24	31	34	38	·52	·38	·06	·03	·01		
21	27	32	32	38	·82	·11	·03	·03	·01	7·864	
22	28	28	33	39	·68	·14	·14	·04	·01		
20	29	32	33	36	1	—	—	—	—	7·884	
24	27	28	31	40	·47	·24	·19	·10	—		
23	23	34	35	35	·48	·48	·02	·01	·31	7·889	
25	25	26	37	37	·35	·35	·27	·01	·01		
20	30	30	35	35	1	—	—	—	—	7·894	
25	25	30	30	40	·38	·38	·12	·12	—		
21	26	32	35	36	·82	·13	·03	·01	·01	7·895	
24	25	28	34	39	·44	·35	·17	·04	·01		
21	27	30	36	36	·82	·11	·05	·01	·01	7·926	
24	24	30	33	39	·42	·42	·10	·05	·01		
22	25	31	35	37	·66	·26	·05	·02	·01	7·938	
23	25	29	35	38	·54	·31	·11	·02	·01		
21	28	29	35	37	·82	·09	·07	·02	·01	8·026	
23	25	31	32	39	·54	·32	·08	·06	·01		
22	27	28	36	37	·68	·17	·13	·02	·01	8·053	
23	24	32	33	38	·52	·38	·05	·04	·01		
22	25	31	36	36	·66	·26	·05	·02	·02	8·055	
24	24	29	35	38	·42	·42	·12	·02	·01		
21	28	30	33	38	·82	·09	·05	·03	·01	8·075	
22	27	30	32	39	·68	·17	·09	·05	·01		
20	29	32	34	35	1	—	—	—	—	8·077	
20	30	31	33	36	1	—	—	—	—		
24	27	29	30	40	·47	·24	·16	·13	—		
25	26	28	31	40	·39	·31	·20	·10	—		
21	26	33	34	36	·82	·13	·02	·02	·01	8·098	
24	26	27	34	39	·45	·28	·22	·04	·01		
22	26	30	34	38	·67	·21	·08	·03	·01	8·124	
21	28	29	36	36	·82	·09	·07	·01	·01	8·142	
24	24	31	32	39	·43	·43	·08	·06	·01		
20	29	33	33	35	1	—	—	—	—	8·169	
20	30	32	32	36	1	—	—	—	—		
24	28	28	30	40	·48	·20	·20	·13	—		
25	27	27	31	40	·39	·25	·25	·11	—		
21	29	29	33	38	·82	·07	·07	·03	·01	8·176	
22	27	31	31	39	·68	·17	·07	·07	·01		
23	24	31	35	37	·52	·39	·06	·02	·01	8·202	
23	25	29	36	37	·54	·31	·11	·02	·01		
21	26	33	35	35	·82	·13	·02	·01	·01	8·202	
25	25	27	34	39	·36	·36	·23	·04	·01		
21	27	31	34	37	·82	·11	·04	·02	·01	8·221	
23	26	29	33	39	·56	·26	·13	·05	·01		
22	25	32	34	37	·66	·26	·04	·03	·01	8·257	
23	26	28	35	38	·55	·25	·16	·03	·01		
20	29	33	34	34	1	—	—	—	—	8·263	
20	31	31	32	36	1	—	—	—	—		
24	28	29	29	40	·48	·20	·16	·16	—		
26	26	27	31	40	·32	·32	·26	·11	—		
20	30	31	34	35	1	—	—	—	—	8·269	
25	26	29	30	40	·39	·31	·17	·13	—		
21	28	31	32	38	·82	·09	·04	·04	·01	8·273	
22	28	29	32	39	·68	·14	·11	·06	·01		
21	26	34	34	35	·82	·13	·02	·02	·01	8·302	
25	26	26	34	39	·37	·29	·29	·04	·01		
23	24	31	36	36	·52	·39	·06	·02	·02	8·320	
24	24	29	36	37	·42	·42	·12	·02	·01		
22	25	33	33	37	·66	·26	·03	·03	·01	8·360	
23	27	27	35	38	·56	·20	·20	·03	·01		
21	27	32	33	37	·82	·11	·03	·03	·01	8·419	
23	27	28	33	39	·56	·21	·17	·05	·01		
22	26	31	33	38	·67	·21	·06	·04	·01	8·436	
22	27	29	34	38	·68	·17	·11	·03	·01		
21	27	31	35	36	·82	·11	·04	·02	·01	8·438	
24	25	29	33	39	·45	·35	·14	·05	·01		
20	30	32	33	35	1	—	—	—	—	8·449	
25	27	28	30	40	·39	·26	·21	·14	—		
21	29	30	32	38	·82	·07	·06	·04	·01	8·470	
22	28	30	31	39	·68	·14	·09	·07	·01		
22	25	32	35	36	·66	·26	·04	·02	·02	8·478	
24	25	28	35	38	·44	·35	·17	·03	·01		
22	26	30	35	37	·67	·21	·08	·02	·01	8·488	
23	25	30	34	38	·54	·32	·10	·04	·01		
23	24	32	34	37	·52	·39	·05	·03	·01	8·520	
23	26	28	36	37	·55	·25	·16	·02	·02		
21	28	30	34	37	·82	·09	·06	·02	·01	8·529	
23	26	30	32	39	·56	·26	·11	·07	·01		
22	26	32	32	38	·67	·21	·05	·05	·01	8·537	
22	28	28	34	38	·68	·14	·14	·03	·01		
20	30	32	34	34	1	—	—	—	—	8·542	
20	31	31	33	35	1	—	—	—	—		
25	27	29	29	40	·40	·26	·17	·17	—		
26	26	28	30	40	·32	·32	·21	·14	—		
21	29	31	31	38	·82	·07	·05	·05	·01	8·565	
22	29	29	31	39	·68	·12	·12	·08	·01		
22	26	30	36	36	·67	·21	·08	·02	·02	8·604	
24	24	30	34	38	·42	·42	·10	·04	·01		
23	24	33	33	37	·52	·38	·04	·04	·01	8·623	
23	27	27	36	37	·56	·20	·20	·02	·02		
21	29	29	34	37	·82	·07	·07	·02	·01	8·629	
23	26	31	31	39	·56	·26	·09	·09	·01		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
20	30	33	33	34	1	—	—	—	—	8·632	
20	31	32	32	35	1	—	—	—	—		
25	28	28	29	40	·40	·21	·21	·17	—		
26	27	27	30	40	·33	·27	·27	·14	—		
20	31	31	34	34	1	—	—	—	—	8·634	
26	26	29	29	40	·32	·32	·18	·18	—		
21	30	30	31	38	·82	·06	·06	·05	·01	8·660	
22	29	30	30	39	·69	·12	·09	·09	·01		
22	25	33	34	36	·66	·26	·04	·03	·02	8·681	
24	26	27	35	38	·45	·28	·22	·03	·01		
21	27	32	34	36	·82	·11	·03	·02	·01	8·731	
24	26	28	33	39	·46	·29	·19	·06	·01		
23	24	32	35	36	·52	·39	·05	·03	·02	8·741	
24	25	28	36	37	·44	·35	·17	·02	·02		
21	28	30	35	36	·82	·09	·06	·02	·01	8·744	
24	25	30	32	39	·45	·35	·12	·07	·01		
22	25	33	35	35	·66	·26	·04	·02	·02	8·785	
25	25	27	35	38	·36	·36	·23	·03	·01		
22	27	29	35	37	·68	·17	·11	·03	·02	8·798	
23	25	31	33	38	·54	·32	·08	·05	·01		
20	31	32	33	34	1	—	—	—	—	8·812	
26	27	28	29	40	·33	·27	·22	·18	—		
21	28	31	33	37	·82	·09	·05	·03	·01	8·818	
23	27	29	32	39	·57	·22	·14	·07	·01		
21	27	33	33	36	·82	·11	·03	·03	·01	8·826	
24	27	27	33	39	·46	·23	·23	·06	·01		
21	27	32	35	35	·52	·11	·03	·02	·02	8·832	
25	25	28	33	39	·37	·37	·19	·06	·01		
22	27	30	33	38	·68	·18	·09	·05	·01	8·838	
21	29	29	35	36	·82	·07	·07	·02	·01	8·844	
24	25	31	31	39	·45	·35	·09	·09	·01		
23	25	30	35	37	·54	·32	·10	·03	·02	8·852	
22	25	34	34	35	·66	·26	·03	·03	·02	8·884	
25	26	26	35	38	·37	·29	·29	·03	·01		
22	26	31	34	37	·67	·21	·07	·03	·02	8·894	
23	26	29	34	38	·56	·26	·13	·04	·01		
22	28	28	35	37	·68	·14	·14	·03	·02	8·898	
23	25	32	32	38	·54	·32	·06	·06	·01		
20	31	33	33	33	1	—	—	—	—	8·900	
20	32	32	32	34	1	—	—	—	—		
26	28	28	28	40	·33	·22	·22	·22	—		
27	27	27	29	40	·27	·27	·27	·18	—		
21	28	32	32	37	·82	·09	·04	·04	·01	8·912	
23	28	28	32	39	·57	·18	·18	·07	·01		
22	27	29	36	36	·68	·17	·11	·02	·02	8·913	
24	24	31	33	38	·43	·43	·08	·05	·01		
23	24	33	34	36	·52	·39	·04	·03	·02	8·944	
24	26	27	36	37	·45	·28	·22	·02	·02		
23	25	30	36	36	·54	·32	·10	·02	·02	8·968	
24	24	30	35	37	·43	·43	·10	·03	·02		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
20	32	32	33	33	1	—	—	—	—	8·988	
27	27	28	28	40	·28	·28	·22	·22	—		
21	29	30	33	37	·82	·07	·06	·03	·01	9·010	
23	27	30	31	39	·57	·22	·11	·09	·01		
22	28	28	36	36	·68	·14	·14	·02	·02	9·013	
24	24	32	32	38	·43	·43	·07	·07	·01		
21	27	33	34	35	·82	·11	·03	·02	·02	9·022	
25	26	27	33	39	·38	·30	·24	·06	·01		
22	27	31	32	38	·68	·18	·07	·06	·01	9·031	
22	28	29	33	38	·68	·14	·11	·05	·01		
23	24	33	35	35	·52	·39	·04	·03	·03	9·047	
25	25	27	36	37	·36	·36	·23	·02	·02		
24	24	30	36	36	·43	·43	·10	·02	·02	9·084	
22	26	32	33	37	·67	·21	·05	·04	·02	9·090	
23	27	28	34	38	·56	·21	·17	·04	·01		
22	26	31	35	36	·67	·21	·07	·03	·02	9·111	
24	25	29	34	38	·45	·35	·14	·04	·01		
21	27	34	34	34	·82	·11	·02	·02	·02	9·117	
26	26	26	33	39	·21	·31	·31	·06	·01		
21	28	31	34	36	·82	·09	·05	·02	·02	9·124	
24	26	29	32	39	·46	·29	·15	·08	·01		
23	24	34	34	35	·52	·39	·03	·03	·03	9·147	
25	26	26	36	37	·37	·29	·29	·03	·02		
21	29	31	32	37	·82	·07	·05	·04	·01	9·194	
23	28	29	31	39	·57	·18	·15	·10	·01		
21	28	31	35	35	·82	·09	·05	·02	·02	9·224	
25	25	29	32	39	·38	·38	·16	·08	·01		
23	25	31	34	37	·54	·32	·08	·04	·02	9·255	
23	26	29	35	37	·56	·26	·13	·03	·02		
21	30	30	32	37	·82	·06	·06	·04	·01	9·286	
23	28	30	30	39	·57	·18	·12	·12	·01		
22	27	30	34	37	·68	·18	·09	·04	·02	9·290	
23	26	30	33	38	·56	·26	·11	·06	·01		
21	28	32	33	36	·82	·09	·04	·03	·02	9·309	·0000
24	27	28	32	39	·47	·24	·20	·08	·01		
21	29	30	34	36	·82	·07	·06	·03	·02	9·314	·0001
24	26	30	31	39	·47	·30	·13	·10	·01		
22	28	30	32	38	·68	·15	·10	·06	·01	9·314	·0001
23	26	29	36	36	·56	·26	·13	·03	·03	9·370	·0001
24	24	31	34	37	·43	·43	·08	·04	·02		
21	30	31	31	37	·83	·06	·05	·05	·01	9·377	·0001
23	29	29	30	39	·57	·15	·15	·12	·01		
22	26	32	34	36	·67	·21	·06	·04	·02	9·400	·0001
24	26	28	34	38	·46	·29	·19	·05	·02		
22	28	31	31	38	·68	·15	·08	·08	·01	9·407	·0001
22	29	29	32	38	·69	·12	·12	·06	·01		
21	29	30	35	35	·82	·07	·06	·02	·02	9·413	·0001
25	25	30	31	39	·38	·38	·13	·11	·01		
23	25	32	33	37	·54	·32	·07	·05	·02	9·450	·0001
23	27	28	35	37	·56	·21	·17	·03	·02		
23	25	31	35	36	·54	·32	·08	·03	·03	9·472	·0001
24	25	29	35	37	·45	·35	·14	·04	·02		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=5$ (*cont.*)											
22	28	29	34	37	·68	·14	·12	·04	·02	9·481	·0001
23	26	31	32	38	·56	·26	·09	·07	·02		
22	26	33	33	36	·67	·21	·04	·04	·02	9·495	·0001
24	27	27	34	38	·47	·24	·24	·05	·02		
21	28	32	34	35	·82	·09	·04	·03	·02	9·500	·0001
25	28	28	32	39	·39	·31	·20	·09	·01		
22	26	32	35	35	·67	·21	·06	·03	·03	9·501	·0001
25	25	28	34	38	·37	·37	·19	·05	·02		
22	27	30	35	36	·68	·18	·09	·03	·02	9·504	·0001
24	25	30	33	38	·45	·35	·12	·06	·02		
23	27	28	36	36	·56	·21	·17	·03	·03	9·564	·0001
24	24	32	33	37	·43	·43	·07	·06	·02		
22	27	31	33	37	·68	·18	·07	·05	·02	9·573	·0001
23	27	29	33	38	·57	·22	·14	·06	·02		
21	29	31	33	36	·82	·08	·05	·03	·02	9·586	·0001
24	27	29	31	39	·47	·25	·16	·11	·01		
24	24	31	35	36	·43	·43	·08	·03	·03	9·587	·0001
24	25	29	36	36	·45	·35	·14	·03	·03		
22	29	30	31	38	·69	·12	·10	·08	·01	9·589	·0001
21	28	33	33	35	·82	·09	·03	·03	·02	9·590	·0001
25	27	27	32	39	·39	·26	·26	·09	·01		
22	27	32	32	37	·68	·18	·06	·06	·02	9·665	·0001
23	28	28	33	38	·57	·18	·18	·06	·02		
21	29	32	32	36	·82	·08	·04	·04	·02	9·675	·0001
24	28	28	31	39	·48	·20	·20	·11	·01		
21	30	30	33	36	·83	·06	·06	·03	·02	9·676	·0001
24	27	30	30	39	·48	·25	·13	·13	·01		
22	30	30	30	38	·69	·10	·10	·10	·01	9·679	·0001
21	28	33	34	34	·82	·09	·03	·03	·03	9·683	·0001
26	26	27	32	39	·32	·32	·26	·09	·01		
22	26	33	34	35	·67	·21	·05	·04	·03	9·690	·0001
25	26	27	34	38	·38	·30	·25	·05	·02		
22	28	29	35	36	·68	·14	·12	·03	·02	9·694	·0001
24	25	31	32	38	·45	·36	·10	·08	·02		
23	26	30	34	37	·56	·26	·11	·05	·02	9·742	·0001
23	25	32	34	36	·54	·32	·07	·04	·03	9·760	·0001
24	26	28	35	37	·46	·29	·19	·04	·02		
21	29	31	34	35	·82	·08	·05	·03	·02	9·775	·0001
25	26	29	31	39	·39	·32	·17	·11	·01		
22	26	34	34	34	·67	·22	·04	·04	·04	9·784	·0001
26	26	26	34	38	·31	·31	·31	·05	·02		
22	28	30	33	37	·68	·15	·10	·05	·02	9·850	·0001
23	27	30	32	38	·57	·22	·12	·08	·02		
21	30	31	32	36	·83	·06	·05	·04	·02	9·853	·0001
24	28	29	30	39	·48	·20	·17	·14	·01		
23	25	33	33	36	·54	·32	·05	·05	·03	9·854	·0001
24	27	27	35	37	·47	·24	·24	·04	·02		
23	25	32	35	35	·54	·32	·07	·03	·03	9·860	·0001
25	25	28	35	37	·37	·37	·19	·04	·02		
21	30	30	34	35	·83	·06	·06	·03	·02	9·865	·0001
25	26	30	30	39	·39	·32	·14	·14	·01		
24	24	32	34	36	·43	·43	·07	·05	·03	9·874	·0002
24	26	28	36	36	·46	·29	·19	·03	·03		
22	27	31	34	36	·68	·18	·08	·04	·03	9·878	·0002
24	26	29	33	38	·47	·29	·16	·07	·02		
21	31	31	31	36	·83	·05	·05	·05	·02	9·940	·0002
24	29	29	29	39	·48	·17	·17	·17	·01		
22	29	29	33	37	·69	·12	·12	·05	·02	9·941	·0002
23	27	31	31	38	·57	·22	·10	·10	·02		
21	29	32	33	35	·82	·08	·04	·03	·02	9·951	·0002
25	27	28	31	39	·40	·26	·21	·12	·01		
23	26	30	35	36	·56	·26	·11	·04	·03	9·955	·0002
24	25	30	34	37	·45	·35	·12	·05	·02		
24	24	33	33	36	·43	·43	·06	·06	·03	9·968	·0002
24	27	27	36	36	·47	·24	·24	·03	·03		
24	24	32	35	35	·43	·43	·07	·04	·04	9·975	·0002
25	25	28	36	36	·37	·37	·19	·03	·03		
22	27	31	35	35	·68	·18	·08	·03	·03	9·977	·0002
25	25	29	33	38	·38	·38	·16	·07	·02		
23	26	31	33	37	·56	·27	·09	·06	·02	10·021	·0002
23	27	29	34	37	·57	·22	·14	·05	·02		
22	28	31	32	37	·69	·15	·08	·07	·02	10·029	·0002
23	28	29	32	38	·57	·18	·15	·08	·02		
21	29	32	34	34	·83	·08	·04	·03	·03	10·041	·0002
26	26	28	31	39	·33	·33	·22	·12	·01		
23	25	33	34	35	·54	·32	·06	·04	·04	10·048	·0002
25	26	27	35	37	·38	·31	·25	·04	·02		
22	27	32	33	36	·68	·18	·06	·05	·03	10·059	·0002
24	27	28	33	38	·47	·24	·20	·07	·02		
23	26	32	32	37	·56	·27	·08	·08	·02	10·112	·0002
23	28	28	34	37	·57	·18	·18	·05	·02		
21	30	31	33	35	·83	·06	·05	·04	·02	10·126	·0002
25	27	29	30	39	·40	·26	·18	·15	·01		
21	29	33	33	34	·83	·08	·03	·03	·03	10·128	·0003
26	27	27	31	39	·33	·27	·27	·12	·01		
23	25	34	34	34	·54	·32	·04	·04	·04	10·142	·0003
26	26	26	35	37	·31	·31	·31	·04	·03		
22	28	30	34	36	·68	·15	·10	·04	·03	10·152	·0003
24	26	30	32	38	·47	·30	·13	·09	·02		
24	24	33	34	35	·43	·43	·06	·05	·04	10·162	·0003
25	26	27	36	36	·38	·31	·25	·03	·03		
24	25	30	35	36	·45	·35	·12	·04	·03	10·168	·0003
22	29	30	32	37	·69	·12	·10	·07	·02	10·206	·0003
23	28	30	31	38	·57	·18	·12	·10	·02		
21	30	32	32	35	·83	·06	·04	·04	·02	10·212	·0003
25	28	28	30	39	·40	·22	·22	·15	·01		
21	30	31	34	34	·83	·06	·05	·03	·03	10·216	·0003
26	26	29	30	39	·33	·33	·18	·15	·01		
23	27	29	35	36	·57	·22	·14	·04	·03	10·232	·0003
24	25	31	33	37	·45	·36	·10	·07	·03		
22	29	29	34	36	·69	·12	·12	·04	·03	10·242	·0003
24	26	31	31	38	·47	·30	·11	·11	·02		
22	27	32	34	35	·68	·18	·06	·04	·03	10·249	·0003
25	26	28	33	38	·39	·31.	·21	·07	·02		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
$n=5$ (*cont.*)											
22	28	30	35	35	·68	·15	·10	·03	·03	10·250	·0003
25	25	30	32	38	·38	·38	·13	·09	·02		
24	24	34	34	34	·43	·43	·05	·05	·05	10·256	·0003
26	26	26	36	36	·31	·31	·31	·03	·03		
22	29	31	31	37	·69	·12	·08	·08	·02	10·294	·0004
23	29	29	31	38	·58	·15	·15	·10	·02		
21	31	31	32	35	·83	·05	·05	·04	·02	10·298	·0004
25	28	29	29	39	·40	·22	·18	·18	·01		
23	28	28	35	36	·57	·18	·18	·04	·03	10·323	·0004
24	25	32	32	37	·45	·36	·08	·08	·03		
23	26	31	34	36	·56	·27	·09	·05	·03	10·325	·0004
24	26	29	34	37	·47	·30	·16	·06	·03		
22	27	33	33	35	·68	·18	·05	·05	·03	10·338	·0004
25	27	27	33	38	·39	·26	·26	·07	·02		
22	29	29	35	35	·69	·12	·12	·04	·04	10·339	·0004
25	25	31	31	38	·38	·38	·11	·11	·02		
22	30	30	31	37	·69	·10	·10	·08	·02	10·380	·0004
23	29	30	30	38	·58	·15	·13	·13	·02		
23	27	30	33	37	·57	·22	·12	·06	·03	10·383	·0004
21	30	32	33	34	·83	·06	·04	·04	·03	10·387	·0004
26	27	28	30	39	·34	·27	·23	·15	·01		
22	28	31	33	36	·69	·15	·08	·05	·03	10·417	·0005
24	27	29	32	38	·48	·25	·17	·09	·02		
23	26	31	35	35	·56	·27	·09	·04	·04	10·423	·0005
25	25	29	34	37	·38	·38	·16	·06	·03		
22	27	33	34	34	·68	·18	·05	·04	·04	10·429	·0005
26	26	27	33	38	·32	·32	·26	·08	·02		
21	31	31	33	34	·83	·05	·05	·04	·03	10·472	·0005
26	27	29	29	39	·34	·28	·19	·19	·01		
21	30	33	33	33	·83	·07	·04	·04	·04	10·473	·0005
27	27	27	30	39	·28	·28	·28	·15	·01		
22	28	32	32	36	·69	·15	·07	·07	·03	10·504	·0005
24	28	28	32	38	·48	·20	·20	·09	·02		
23	26	32	33	36	·56	·27	·08	·06	·03	10·504	·0005
24	27	28	34	37	·47	·24	·20	·06	·03		
24	25	31	34	36	·45	·36	·10	·05	·03	10·536	·0006
24	26	29	35	36	·47	·30	·16	·04	·04		
21	31	32	32	34	·83	·05	·04	·04	·03	10·556	·0006
26	28	28	29	39	·34	·23	·23	·19	·01		
23	27	31	32	37	·57	·22	·10	·08	·03	10·559	·0006
23	28	29	33	37	·57	·18	·15	·07	·03		
22	29	30	33	36	·69	·12	·10	·06	·03	10·591	·0006
24	27	30	31	38	·48	·25	·14	·11	·02		
22	28	31	34	35	·69	·15	·08	·05	·04	10·604	·0007
25	26	29	32	38	·39	·32	·17	·10	·02		
24	25	31	35	35	·45	·36	·10	·04	·04	10·634	·0007
25	25	29	35	36	·38	·38	·16	·05	·04		
21	31	32	33	33	·83	·05	·04	·04	·04	10·641	·0007
27	27	28	29	39	·28	·28	·23	·19	·01		
23	27	30	34	36	·57	·22	·12	·05	·03	10·683	·0007
24	26	30	33	37	·47	·30	·13	·07	·03		
23	26	32	34	35	·56	·27	·08	·05	·04	10·693	·0008
25	26	28	34	37	·39	·31	·21	·06	·03		
$n=5$ (*cont.*)											
24	25	32	33	36	·45	·36	·08	·07	·04	10·715	·0008
24	27	28	35	36	·47	·24	·20	·05	·04		
21	32	32	32	33	·83	·05	·05	·05	·04	10·724	·0008
27	28	28	28	39	·28	·23	·23	·23	·01		
22	29	31	32	36	·69	·13	·09	·07	·03	10·762	·0008
24	28	29	31	38	·48	·21	·17	·12	·02		
22	28	32	33	35	·69	·15	·07	·06	·04	10·776	·0009
25	27	28	32	38	·40	·26	·22	·10	·02		
22	29	30	34	35	·69	·12	·10	·05	·04	10·777	·0009
25	26	30	31	38	·39	·32	·14	·12	·02		
23	27	30	35	35	·57	·22	·12	·04	·04	10·780	·0010
25	25	30	33	37	·38	·38	·14	·07	·03		
23	26	33	33	35	·56	·27	·06	·06	·04	10·781	·0010
25	27	27	34	37	·39	·26	·26	·06	·03		
23	28	30	32	37	·57	·19	·13	·09	·03	10·818	·0010
22	30	30	32	36	·69	·10	·10	·07	·03	10·846	·0010
24	28	30	30	38	·48	·21	·14	·14	·02		
23	28	29	34	36	·57	·18	·15	·06	·04	10·857	·0011
24	26	31	32	37	·47	·30	·11	·09	·03		
22	28	32	34	34	·69	·15	·07	·05	·05	10·866	·0011
26	26	28	32	38	·33	·33	·22	·10	·02		
23	26	33	34	34	·56	·27	·06	·05	·05	10·872	·0011
26	26	27	34	37	·32	·32	·26	·06	·03		
23	28	31	31	37	·58	·19	·10	·10	·03	10·903	·0011
23	29	29	32	37	·58	·15	·15	·09	·03		
24	25	32	34	35	·45	·36	·08	·06	·05	10·903	·0012
25	26	28	35	36	·39	·31	·21	·05	·04		
22	30	31	31	36	·69	·11	·09	·09	·03	10·930	·0012
24	29	29	30	38	·49	·17	·17	·14	·02		
23	27	31	33	36	·57	·22	·10	·07	·04	10·944	·0013
24	27	29	33	37	·48	·25	·17	·08	·03		
22	28	33	33	34	·69	·15	·06	·06	·05	10·952	·0013
26	27	27	32	38	·33	·27	·27	·10	·02		
23	28	29	35	35	·57	·18	·15	·05	·05	10·954	·0013
25	25	31	32	37	·38	·38	·11	·09	·03		
24	26	30	34	36	·47	·30	·13	·06	·04	10·982	·0014
24	25	33	33	35	·46	·36	·07	·07	·05	10·991	·0014
25	27	27	35	36	·39	·26	·26	·05	·04		
23	27	32	32	36	·57	·23	·08	·08	·04	11·029	·0014
24	28	28	33	37	·48	·21	·21	·08	·03		
22	29	31	33	35	·69	·13	·09	·06	·04	11·031	·0015
25	27	29	31	38	·40	·27	·18	·12	·02		
23	29	30	31	37	·58	·16	·13	·11	·03	11·071	·0016
24	26	30	35	35	·47	·30	·13	·05	·05	11·079	·0016
25	25	30	34	36	·38	·38	·14	·06	·04		
24	25	33	34	34	·46	·36	·07	·06	·06	11·082	·0016
26	26	27	35	36	·32	·32	·26	·05	·04		
22	29	32	32	35	·69	·13	·07	·07	·04	11·115	·0017
22	30	30	33	35	·69	·11	·11	·06	·04		
25	27	30	30	38	·40	·27	·15	·15	·02		
25	28	28	31	38	·41	·22	·22	·13	·02		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
22	29	31	34	34	·69	·13	·09	·05	·05	11·119	·0018
26	26	29	31	38	·33	·33	·19	·13	·02		
23	27	31	34	35	·57	·23	·10	·06	·05	11·129	·0018
25	26	29	33	37	·39	·32	·18	·08	·03		
23	30	30	30	37	·58	·13	·13	·13	·03	11·154	·0019
25	25	30	35	35	·38	·38	·14	·05	·05	11·176	·0019
23	28	30	33	36	·58	·19	·13	·07	·04	11·199	·0020
24	27	30	32	37	·48	·25	·14	·10	·03		
22	30	30	34	34	·69	·11	·11	·05	·05	11·202	·0020
26	26	30	30	38	·33	·33	·15	·15	·02		
24	26	31	33	36	·47	·30	·11	·08	·04	11·241	·0021
24	27	29	34	36	·48	·25	·17	·06	·04		
22	30	31	32	35	·69	·11	·09	·07	·04	11·280	·0022
25	28	29	30	38	·41	·23	·19	·15	·02		
23	29	29	33	36	·58	·16	·16	·07	·04	11·283	·0023
24	27	31	31	37	·48	·25	·12	·12	·03		
22	29	32	33	34	·69	·13	·07	·06	·05	11·286	·0024
26	27	28	31	38	·34	·28	·23	·13	·02		
23	27	32	33	35	·57	·23	·09	·07	·05	11·300	·0025
25	27	28	33	37	·40	·27	·22	·08	·03		
24	26	32	32	36	·47	·30	·09	·09	·04	11·326	·0026
24	28	28	34	36	·48	·21	·21	·06	·04		
24	27	29	35	35	·48	·25	·17	·05	·05	11·337	·0027
25	25	31	33	36	·38	·38	·12	·08	·04		
22	31	31	31	35	·69	·09	·09	·09	·04	11·362	·0027
25	29	29	29	38	·41	·19	·19	·19	·02		
23	28	31	32	36	·58	·19	·11	·09	·04	11·366	·0028
24	28	29	32	37	·48	·21	·17	·10	·03		
22	29	33	33	33	·69	·13	·06	·06	·06	11·370	·0028
27	27	27	31	38	·28	·28	·28	·13	·02		
23	28	30	34	35	·58	·19	·13	·06	·05	11·383	·0030
25	26	30	32	37	·40	·32	·15	·10	·03		
23	27	32	34	34	·57	·23	·09	·06	·06	11·388	·0031
26	26	28	33	37	·33	·33	·22	·08	·03		
24	28	28	35	35	·48	·21	·21	·05	·05	11·422	·0031
25	25	32	32	36	·38	·38	·10	·10	·04		
24	26	31	34	35	·47	·30	·11	·06	·05	11·426	·0033
25	26	29	34	36	·39	·32	·18	·07	·04		
22	30	31	33	34	·69	·11	·09	·06	·05	11·450	·0034
26	27	29	30	38	·34	·28	·19	·16	·02		
23	29	29	34	35	·58	·16	·16	·06	·05	11·466	·0035
25	26	31	31	37	·40	·32	·12	·12	·03		
23	27	33	33	34	·57	·23	·07	·07	·06	11·473	·0036
26	27	27	33	37	·33	·27	·27	·09	·03		
25	25	31	34	35	·38	·38	·12	·06	·05	11·522	·0037
25	26	29	35	35	·39	·32	·18	·05	·05		
23	29	30	32	36	·58	·16	·13	·09	·04	11·530	·0039
24	28	30	31	37	·49	·21	·15	·12	·03		
22	30	32	32	34	·69	·11	·07	·07	·05	11·531	·0040
26	28	28	30	38	·34	·23	·23	·16	·03		
24	27	30	33	36	·48	·25	·14	·08	·04	11·578	·0041
24	26	32	33	35	·47	·30	·09	·08	·05	11·595	·0043
25	27	28	34	36	·40	·27	·22	·07	·04		
23	29	31	31	36	·58	·16	·11	·11	·04	11·611	·0045
24	29	29	31	37	·49	·18	·18	·12	·03		
22	31	31	32	34	·69	·09	·09	·08	·05	11·612	·0046
26	28	29	29	38	·35	·24	·20	·20	·03		
22	30	32	33	33	·69	·11	·08	·06	·06	11·614	·0047
27	27	28	30	38	·29	·29	·24	·16	·03		
23	28	31	33	35	·58	·19	·11	·07	·05	11·632	·0049
25	27	29	32	37	·40	·27	·19	·11	·03		
24	26	32	34	34	·47	·30	·10	·06	·06	11·683	·0051
26	26	28	34	36	·33	·33	·22	·07	·05		
25	25	32	33	35	·38	·38	·10	·08	·05	11·691	·0052
25	27	28	35	35	·40	·27	·22	·06	·06		
23	30	30	31	36	·58	·13	·13	·11	·04	11·692	·0053
24	29	30	30	37	·49	·18	·15	·15	·03		
22	31	31	33	33	·69	·09	·09	·06	·06	11·694	·0054
27	27	29	29	38	·29	·29	·20	·20	·03		
23	28	32	32	35	·58	·19	·09	·09	·05	11·714	·0056
25	28	28	32	37	·41	·23	·23	·11	·04		
23	28	31	34	34	·58	·19	·11	·06	·06	11·719	·0057
26	26	29	32	37	·33	·33	·19	·11	·04		
24	27	31	32	36	·48	·26	·12	·10	·04	11·742	·0060
24	28	29	33	36	·48	·21	·18	·08	·04		
24	27	30	34	35	·48	·25	·14	·07	·05	11·761	·0063
25	26	30	33	36	·40	·32	·15	·08	·04		
24	26	33	33	34	·47	·30	·08	·08	·07	11·767	·0065
26	27	27	34	36	·33	·27	·27	·07	·05		
22	31	32	32	33	·69	·09	·08	·08	·06	11·774	·0067
27	28	28	29	38	·29	·24	·24	·20	·03		
25	25	32	34	34	·38	·38	·10	·07	·07	11·779	·0067
26	26	28	35	35	·33	·33	·22	·06	·06		
25	29	30	33	35	·58	·16	·13	·08	·05	11·795	·0071
25	27	30	31	37	·41	·27	·16	·13	·04		
22	32	32	32	32	·69	·08	·08	·08	·08	11·855	·0071
28	28	28	28	38	·24	·24	·24	·24	·03		
25	25	33	33	34	·38	·38	·08	·08	·07	11·862	·0072
26	27	27	35	35	·33	·27	·27	·06	·06		
23	29	30	34	34	·58	·16	·13	·06	·06	11·881	·0074
26	26	30	31	37	·34	·34	·16	·13	·04		
23	28	32	33	34	·58	·19	·09	·08	·06	11·884	·0078
26	27	28	32	37	·34	·28	·23	·11	·04		
24	28	29	34	35	·49	·21	·18	·07	·06	11·924	·0083
25	26	31	32	36	·40	·32	·13	·10	·05		
25	26	30	34	35	·40	·32	·15	·07	·06	11·943	·0085
23	29	31	32	35	·58	·16	·11	·09	·05	11·956	·0090
25	28	29	31	37	·41	·23	·19	·13	·04		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
n = 5 (*cont.*)											
23	28	33	33	33	·58	·19	·08	·08	·08	11·967	·0091
27	27	27	32	37	·28	·28	·28	·11	·04		
24	28	30	32	36	·49	·22	·15	·10	·05	11·986	·0093
24	27	31	33	35	·48	·26	·12	·08	·06	12·007	·0098
25	27	29	33	36	·41	·27	·19	·09	·05		
23	30	30	32	35	·58	·14	·14	·09	·05	12·035	·0101
25	28	30	30	37	·41	·23	·16	·16	·04		
24	28	31	31	36	·49	·22	·12	·12	·05	12·065	·0104
24	29	29	32	36	·49	·18	·18	·10	·05		
24	27	32	32	35	·48	·26	·10	·10	·06	12·088	·0107
25	28	28	33	36	·41	·23	·23	·09	·05		
24	27	31	34	34	·48	·26	·12	·07	·07	12·094	·0111
26	26	29	33	36	·34	·34	·19	·09	·05		
23	30	31	31	35	·58	·14	·11	·11	·05	12·114	·0114
25	29	29	30	37	·41	·19	·19	·16	·04		
23	29	31	33	34	·58	·16	·11	·08	·06	12·124	·0121
26	27	29	31	37	·34	·28	·20	·14	·04		
25	26	31	33	35	·40	·33	·13	·09	·06	12·188	·0128
25	27	29	34	35	·41	·27	·19	·07	·06		
23	30	30	33	34	·58	·14	·14	·08	·07	12·203	·0132
26	27	30	30	37	·35	·29	·17	·17	·04		
23	29	32	32	34	·58	·16	·09	·09	·06	12·203	·0136
26	28	28	31	37	·35	·24	·24	·14	·04		
24	29	30	31	36	·49	·18	·15	·13	·05	12·224	·0141
24	28	30	33	35	·49	·22	·15	·09	·06	12·248	·0149
25	27	30	32	36	·41	·27	·16	·11	·05		
24	27	32	33	34	·48	·26	·10	·09	·07	12·257	·0158
26	27	28	33	36	·34	·28	·23	·09	·05		
25	26	32	32	35	·40	·33	·11	·11	·06	12·269	·0163
25	28	28	34	35	·41	·23	·23	·07	·06		
25	26	31	34	34	·40	·33	·13	·07	·07	12·274	·0167
26	26	29	34	35	·34	·34	·19	·08	·06		
23	29	32	33	33	·58	·16	·10	·08	·08	12·284	·0172
27	27	28	31	37	·29	·29	·24	·14	·04		
24	30	30	30	36	·49	·15	·15	·15	·05	12·302	·0173
24	29	29	33	35	·49	·18	·18	·09	·06	12·327	·0178
25	27	31	31	36	·41	·28	·13	·13	·05		
24	28	30	34	34	·49	·22	·15	·07	·07	12·334	·0183
26	26	30	32	36	·34	·34	·16	·11	·05		
24	27	33	33	33	·48	·26	·09	·09	·09	12·338	·0185
27	27	27	33	36	·28	·28	·28	·09	·05		
23	30	31	32	34	·58	·14	·12	·10	·07	12·360	·0196
26	28	29	30	37	·35	·24	·20	·17	·04		
24	28	31	32	35	·49	·22	·13	·11	·06	12·406	·0209
25	28	29	32	36	·41	·23	·19	·11	·05		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
n = 5 (*cont.*)											
24	29	29	34	34	·49	·18	·18	·07	·07	12·412	·0212
26	26	31	31	36	·34	·34	·14	·14	·05		
25	26	32	33	34	·40	·33	·11	·09	·07	12·437	·0225
26	27	28	34	35	·34	·28	·23	·08	·06		
23	31	31	31	34	·58	·12	·12	·12	·07	12·438	·0227
26	29	29	29	37	·35	·20	·20	·20	·04		
23	30	31	33	33	·58	·14	·12	·08	·08	12·441	·0233
27	27	29	30	37	·29	·29	·20	·17	·04		
25	27	30	33	35	·41	·27	·16	·09	·06	12·509	·0241
25	26	33	33	33	·40	·33	·09	·09	·09	12·518	·0244
27	27	27	34	35	·29	·29	·29	·08	·06		
23	30	32	32	33	·59	·14	·10	·10	·08	12·519	·0251
27	28	28	30	37	·29	·25	·25	·17	·04		
24	29	30	32	35	·49	·18	·15	·11	·06	12·562	·0268
25	28	30	31	36	·42	·23	·16	·14	·05		
24	28	31	33	34	·49	·22	·13	·09	·07	12·572	·0286
26	27	29	32	36	·35	·29	·20	·12	·05		
25	27	30	34	34	·41	·28	·16	·08	·08	12·594	·0294
26	26	30	33	35	·34	·34	·16	·09	·06		
23	31	31	32	33	·59	·12	·12	·10	·08	12·596	·0304
27	28	29	29	37	·30	·25	·21	·21	·04		
24	29	31	31	35	·49	·19	·13	·13	·06	12·640	·0314
25	29	29	31	36	·42	·20	·20	·14	·05		
24	28	32	32	34	·49	·22	·11	·11	·07	12·651	·0324
26	28	28	32	36	·35	·24	·24	·12	·05		
25	27	31	32	35	·41	·28	·14	·11	·06	12·666	·0345
25	28	29	33	35	·42	·23	·19	·10	·06		
23	31	32	32	32	·59	·12	·10	·10	·10	12·674	·0348
28	28	28	29	37	·25	·25	·25	·21	·04		
26	26	30	34	34	·34	·34	·16	·08	·08	12·679	·0351
24	30	30	31	35	·49	·16	·16	·13	·06	12·716	·0363
25	29	30	30	36	·42	·20	·17	·17	·05		
24	29	30	33	34	·49	·19	·16	·09	·08	12·727	·0387
26	27	30	31	36	·35	·29	·17	·14	·05		
24	28	32	33	33	·49	·22	·11	·09	·09	12·731	·0399
27	27	28	32	36	·29	·29	·24	·12	·05		
25	28	29	34	34	·41	·23	·19	·08	·08	12·750	·0411
26	26	31	32	35	·34	·34	·14	·12	·06		
25	27	31	33	34	·41	·28	·14	·09	·08	12·831	·0441
26	27	29	33	35	·35	·29	·20	·10	·07		
24	29	31	32	34	·50	·19	·13	·11	·08	12·882	·0474
26	28	29	31	36	·35	·24	·20	·14	·05		
25	28	30	32	35	·42	·23	·17	·12	·07	12·898	·0491
25	27	32	32	34	·41	·28	·11	·11	·08	12·909	·0508
26	28	28	33	35	·35	·24	·24	·10	·07		
26	26	31	33	34	·34	·34	·14	·10	·08	12·915	·0526
26	27	29	34	34	·35	·29	·20	·08	·08		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
24	30	30	32	34	·50	·16	·16	·11	·08	12·958	·0545
26	28	30	30	36	·35	·25	·17	·17	·06		
24	29	31	33	33	·50	·19	·13	·09	·09	12·961	·0565
27	27	29	31	36	·30	·30	·21	·15	·06		
25	28	31	31	35	·42	·24	·14	·14	·07	12·974	·0585
25	29	29	32	35	·42	·20	·20	·12	·07		
25	27	32	33	33	·41	·28	·12	·10	·10	12·988	·0605
27	27	28	33	35	·29	·29	·24	·10	·07		
26	26	32	32	34	·34	·34	·12	·12	·08	12·993	·0616
26	28	28	34	34	·35	·24	·24	·08	·08		
24	30	31	31	34	·50	·16	·13	·13	·08	13·034	·0638
26	29	29	30	36	·36	·21	·21	·17	·06		
24	30	30	33	33	·50	·16	·16	·09	·09	13·037	·0650
27	27	30	30	36	·30	·30	·18	·18	·06		
24	29	32	32	33	·50	·19	·11	·11	·09	13·038	·0673
27	28	28	31	36	·30	·25	·25	·15	·06		
25	28	30	33	34	·42	·24	·17	·10	·08	13·061	·0721
26	27	30	32	35	·35	·29	·17	·12	·07		
26	26	32	33	33	·34	·34	·12	·10	·10	13·073	·0733
27	27	28	34	34	·29	·29	·25	·08	·08		
25	29	30	31	35	·42	·20	·17	·14	·07	13·126	·0761
25	29	29	33	34	·42	·20	·20	·10	·08	13·137	·0789
26	27	31	31	35	·35	·29	·14	·14	·07		
24	30	31	32	33	·50	·16	·13	·11	·09	13·189	·0852
27	28	29	30	36	·30	·25	·21	·18	·06		
25	30	30	30	35	·42	·17	·17	·17	·07	13·201	·0857
25	28	31	32	34	·42	·24	·14	·12	·08	13·214	·0923
26	28	29	32	35	·35	·25	·21	·12	·07		
26	27	30	33	34	·35	·29	·17	·10	·08	13·225	·0957
24	31	31	31	33	·50	·14	·14	·14	·10	13·264	·0969
27	29	29	29	36	·30	·21	·21	·21	·06		
24	30	32	32	32	·50	·16	·11	·11	·11	13·265	·0982
28	28	28	30	36	·25	·25	·25	·18	·06		
25	28	31	33	33	·42	·24	·14	·10	·10	13·293	·1021
27	27	29	32	35	·30	·30	·21	·12	·07		
24	31	31	32	32	·50	·14	·14	·11	·11	13·340	·1042
28	28	29	29	36	·26	·26	·22	·22	·06		
25	29	30	32	34	·42	·20	·17	·12	·08	13·364	·1133
26	28	30	31	35	·36	·25	·18	·15	·07		
25	28	32	32	33	·42	·24	·12	·12	·10	13·369	·1179
27	28	28	32	35	·30	·25	·25	·13	·07		
26	27	31	32	34	·35	·29	·15	·12	·09	13·376	·1272
26	28	29	33	34	·36	·25	·21	·10	·09		
25	29	31	31	34	·42	·20	·14	·14	·08	13·439	·1325
26	29	29	31	35	·36	·21	·21	·15	·07		
25	29	30	33	33	·42	·20	·17	·10	·10	13·443	·1379
27	27	30	31	35	·30	·30	·18	·15	·07		
26	27	31	33	33	·35	·29	·15	·10	·10	13·455	·1433
27	27	29	33	34	·30	·30	·21	·11	·09		
25	30	30	31	34	·42	·17	·17	·15	·09	13·513	·1495
26	29	30	30	35	·36	·21	·18	·18	·07		
26	27	32	32	33	·35	·29	·12	·12	·10	13·531	·1560
27	28	28	33	34	·30	·25	·25	·11	·09		
25	29	31	32	33	·42	·20	·15	·12	·10	13·593	·1707
27	28	29	31	35	·30	·25	·21	·15	·07		
26	28	30	32	34	·36	·25	·18	·13	·09	13·601	·1781
25	30	30	32	33	·43	·17	·17	·12	·10	13·666	·1867
27	28	30	30	35	·31	·26	·18	·18	·07		
25	29	32	32	32	·43	·21	·12	·12	·12	13·668	·1896
28	28	28	31	35	·26	·26	·26	·15	·07		
26	28	31	31	34	·36	·25	·15	·15	·09	13·675	·1983
26	29	29	32	34	·36	·21	·21	·13	·09		
26	28	30	33	33	·36	·25	·18	·11	·11	13·679	·2071
27	27	30	32	34	·30	·30	·18	·13	·09		
25	30	31	31	33	·43	·17	·15	·15	·10	13·740	·2171
27	29	29	30	35	·31	·22	·22	·18	·08		
26	29	29	33	33	·36	·21	·21	·11	·11	13·753	·2222
27	27	31	31	34	·30	·30	·15	·15	·09		
27	27	30	33	33	·30	·30	·18	·11	·11	13·757	·2248
25	30	31	32	32	·43	·18	·15	·12	·12	13·815	·2366
28	28	29	30	35	·26	·26	·22	·19	·08		
26	29	30	31	34	·36	·21	·18	·15	·09	13·822	·2485
26	28	31	32	33	·36	·25	·15	·13	·11	13·828	·2725
27	28	29	32	34	·31	·26	·22	·13	·09		
25	31	31	31	32	·43	·15	·15	·15	·13	13·888	·2771
28	29	29	29	35	·26	·22	·22	·22	·08		
26	30	30	30	34	·36	·18	·18	·18	·09	13·895	·2794
26	28	32	32	32	·36	·25	·13	·13	·13	13·902	·2841
28	28	28	32	34	·26	·26	·26	·13	·09		
27	27	31	32	33	·30	·30	·15	·13	·11	13·905	·2982
27	28	29	33	33	·31	·26	·22	·11	·11		
26	29	30	32	33	·36	·22	·18	·13	·11	13·974	·3310
27	28	30	31	34	·31	·26	·18	·16	·09		
27	27	32	32	32	·30	·30	·13	·13	·13	13·980	·3338
28	28	28	33	33	·26	·26	·26	·11	·11		
26	29	31	31	33	·36	·22	·15	·15	·11	14·047	·3529
27	29	29	31	34	·31	·22	·22	·16	·09		
26	30	30	31	33	·37	·18	·18	·16	·11	14·119	·3751
27	29	30	30	34	·31	·22	·19	·19	·09		
26	29	31	32	32	·37	·22	·16	·13	·13	14·121	·3975
28	28	29	31	34	·26	·26	22	·16	·09		

Σr_1	Σr_2	Σr_3	Σr_4	Σr_5	p_1	p_2	p_3	p_4	p_5	B_1	P
					$n=5$ (*cont.*)						
27	28	30	32	33	·31	·26	·19	·13	·11	14·125	·4200
26	30	30	32	32	·37	·18	·18	·13	·13	14·193	·4330
28	28	30	30	34	·26	·26	·19	·19	·10		
27	28	31	31	33	·31	·26	·16	·16	·11	14·197	·4593
27	29	29	32	33	·31	·22	·22	·13	·11		
26	30	31	31	32	·37	·19	·16	·16	·13	14·265	·4896
28	29	29	30	34	·27	·22	·22	·19	·10		
27	28	31	32	32	·31	·26	·16	·13	·13	14·271	·5203
28	28	29	32	33	·26	·26	·22	·14	·11		
26	31	31	31	31	·37	·16	·16	·16	·16	14·337	·5232
29	29	29	29	34	·23	·23	·23	·23	·10		
27	29	30	31	33	·31	·22	·19	·16	·11	14·341	·5588
27	30	30	30	33	·31	·19	·19	·19	·12	14·412	·5657
27	29	30	32	32	·31	·22	·19	·14	·14	14·414	·6072
28	28	30	31	33	·27	·27	·19	·16	·12		
27	29	31	31	32	·31	·22	·16	·16	·14	14·486	·6556
28	29	29	31	33	·27	·23	·23	·16	·12		
28	28	30	32	32	·27	·27	·19	·14	·14	14·487	·6677
27	30	30	31	32	·32	·19	·19	·16	·14	14·557	·7239
28	29	30	30	33	·27	·23	·19	·19	·12		
28	28	31	31	32	·27	·27	·16	·16	·14	14·559	·7521
28	29	29	32	32	·27	·23	·23	·14	·14		
27	30	31	31	31	·32	·19	·16	·16	·16	14·628	·7739
29	29	29	30	33	·23	·23	·23	·19	·12		
28	29	30	31	32	·27	·23	·19	·17	·14	14·700	·8501
28	30	30	30	32	·27	·20	·20	·20	·14	14·771	·8649
28	29	31	31	31	·27	·23	·17	·17	·17	14·771	·8944
29	29	29	31	32	·23	·23	·23	·17	·14		
28	30	30	31	31	·27	·20	·20	·17	·17	14·842	·9458
29	29	30	30	32	·23	·23	·20	·20	·14		
29	29	30	31	31	·23	·23	·20	·17	·17	14·912	·9756
29	30	30	30	31	·23	·20	·20	·20	·17	14·982	·9987
30	30	30	30	30	·20	·20	·20	·20	·20	15·052	1·0000

MISCELLANEA

Approximate formulae for the percentage points and the probability integral of the non-central χ^2 distribution

By S. H. ABDEL-ATY

University College, London

1. The non-central χ^2 with f degrees of freedom and the parameter λ has a probability-density function which may be written as

$$p(\chi'^2) = \frac{e^{-\frac{1}{2}\chi'^2} e^{-\frac{1}{2}\lambda}}{2^{\frac{1}{2}f}} \sum_{j=0}^{\infty} \frac{(\chi'^2)^{\frac{1}{2}f+j-1}\,\lambda^j}{2^{2j}\,j!\,\Gamma(\frac{1}{2}f+j)},$$

the sth cumulant being

$$\kappa_s = 2^{s-1}(s-1)!\,(f+s\lambda).$$

Some exact values for the probability integral have been given by R. A. Fisher (1928) and by F. Garwood (1934). Tables for use in connexion with the power function of χ^2 have been computed by E. Fix (1949), and several methods of approximation were considered by P. B. Patnaik (1949). It seemed of interest to investigate the possibility of another type of approximation based on normalizing the χ'^2 distribution by means of the cube-root transformation, and thus achieve a rapid method of calculating the probability integral or percentage point when the value required was not already tabulated. The cube-root transformation of Wilson and Hilfertz is so very useful for the central χ^2 that it would not be surprising to find it equally useful in the non-central case.

2. We make the transformation

$$y = \left(\frac{\chi'^2}{f+\lambda}\right)^h,$$

writing

$$r = f+\lambda \quad \text{and} \quad b = \frac{\lambda}{f+\lambda},$$

we find after expansion that $h = \frac{1}{3}$ is the appropriate value to choose for minimizing the coefficient of r^{-2} in $\kappa_2(y)$ and reducing the higher cumulants, $\kappa_s(y)$ for $s>2$, to a minimum.

The moments of y were obtained generally. When h is put equal to $\frac{1}{3}$ we have approximately

$$\kappa_1(y) = 1 - \frac{2}{3^2}r^{-1}(1+b) - \frac{40}{3^4}r^{-2}b^2 + \frac{80}{3^7}r^{-3}(1+3b+33b^2-77b^3) + \frac{176}{3^9}r^{-4}(1+4b-210b^2+2380b^3-2975b^4) + O(r^{-5}),$$

$$\kappa_2(y) = \frac{2}{3^2}r^{-1}(1+b) + \frac{16}{3^3}r^{-2}b^2 - \frac{8}{3^7}r^{-3}(13+39b+405b^2-1025b^3) - \frac{160}{3^8}r^{-4}(1+4b-87b^2+1168b^3-1544b^4) + O(r^{-5}),$$

$$\kappa_3(y) = -\frac{8}{3^3}r^{-2}b^2 + \frac{32}{3^6}r^{-3}(1+3b+21b^2-62b^3) + \frac{32}{3^8}r^{-4}(8+32b-177b^2+4550b^3-6625b^4) + O(r^{-5}),$$

$$\kappa_4(y) = -\frac{16}{3^6}r^{-3}(1+3b+12b^2-44b^3) - \frac{256}{3^8}r^{-4}(1+4b+6b^2+274b^3-458b^4) + O(r^{-5}),$$

$$\kappa_5(y) = \frac{128}{3^8}r^{-4}(1+4b+15b^2+130b^3-260b^4) + O(r^{-5}).$$

Measures of skewness and kurtosis are

$$\beta_1(y) = \frac{8b^4}{r(1+b)^3} - \frac{64}{3^3}\frac{b^2(1+4b+24b^2-41b^3-35b^4)}{r^2(1+b)^4} + \frac{128}{3^6(1+b)^5}O\left(\frac{1}{r^3}\right),$$

$$\beta_2(y) = 3 - \frac{4}{3^2}\frac{(1+3b+12b^2-44b^3)}{r(1+b)^2} - \frac{64}{3^4}\frac{(1+5b+7b^2+271b^3-220b^4-326b^5)}{r^2(1+b)^3}.$$

These values for β_1 and β_2 may be compared with the following values for χ'^2, noting that $0<b<1$:

$$\beta_1(\chi'^2)=\frac{8(1+2b)^2}{r(1+b)^3} \quad \text{and} \quad \beta_2(\chi'^2)=3+\frac{12(1+3b)}{r(1+b)^2}.$$

They indicate that the distribution of $(\chi'^2/r)^{\frac{1}{3}}$ tends to normality more rapidly than that of χ'^2.

As a first approximation we take $(\chi'^2/r)^{\frac{1}{3}}$ to be approximately normally distributed about a mean $1-\frac{2}{9}(1+b)/r$ with variance $\frac{2}{9}(1+b)/r$. We shall refer to this as the 'first approx.' in the numerical results.

3. The Fisher-Cornish (1937) expansion for the standardized deviate of a variable can be used with advantage in our case, since the effect of the cube-root transformation is to make the transformed variable approximately normally distributed.

Writing $$Y=\{y-\kappa_1(y)\}/\sqrt{\{\kappa_2(y)\}},$$

then the 100α % point of Y is given by

$$Y=\xi-r^{-\frac{1}{2}}(1+b)^{-\frac{1}{2}}b^2B_{1.1}-r^{-1}(1+b)^{-3}(LB_{2.1}+b^4B_{2.2})+r^{-\frac{3}{2}}(1+b)^{-\frac{3}{2}}(MB_{3.1}+NB_{3.2}-Lb^2B_{3.3}-b^6B_{3.4}),$$

where $$\alpha=\frac{1}{\sqrt{(2\pi)}}\int_{\xi}^{\infty}e^{-\frac{1}{2}u^2}du.$$

L, M and N are the polynomials in b given by

$$L=(1+b)(1+3b+12b^2-44b^3),$$
$$M=(1+b)^2(1+4b+24b^2-41b^3-35b^4),$$
$$N=(1+b)^2(1+4b+15b^2+130b^3-260b^4),$$

and the B's are the following polynomials in ξ:

$$B_{1.1}=\frac{\sqrt{2}}{3}(\xi^2-1),$$

$$B_{2.1}=\tfrac{1}{54}(\xi^3-3\xi), \qquad B_{2.2}=\tfrac{2}{9}(2\xi^3-5\xi),$$

$$B_{3.1}=\frac{4\sqrt{2}}{81}(\xi^2-1), \qquad B_{3.2}=\frac{2\sqrt{2}}{405}(\xi^4-6\xi^2+3),$$

$$B_{3.3}=\frac{\sqrt{2}}{27}(\xi^4-5\xi^2+2), \qquad B_{3.4}=\frac{4\sqrt{2}}{81}(12\xi^4-53\xi^2+17).$$

These polynomials are tabulated for some values of α in Table 1. The probability integral $F(Y)$ up to a given point can be obtained by a straightforward application of the Edgeworth expansion, thus giving

$$F(Y)=\int_{-\infty}^{Y}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}u^2}du+\left\{\frac{\sqrt{2}}{3}r^{-\frac{1}{2}}(1+b)^{-\frac{1}{2}}b^2H_2+\tfrac{1}{54}r^{-1}(1+b)^{-3}(LH_3-6b^4H_5)\right.$$
$$\left.-\frac{\sqrt{2}}{810}r^{-\frac{3}{2}}(1+b)^{-\frac{3}{2}}(40MH_2+4NH_4+5Lb^2H_6-10b^6H_8)\right\}\left\{\frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}Y^2}\right\},$$

where the H's are the Hermite's polynomials in Y; L, M and N are the same polynomials in b as given above.

In the numerical results, we shall refer to results given by formulae of this section as the 'closer approx.'.

Table 1. *Giving the B-polynomials for some values of* α

α	ξ	$B_{1.1}$	$B_{2.1}$	$B_{2.2}$	$B_{3.1}$	$B_{3.2}$	$B_{3.3}$	$B_{3.4}$
0·5	0·00000	−0·47140	0·00000	0·00000	−0·06984	0·02095	0·10476	1·18724
0·4	0·25335	−0·44115	−0·01377	−0·27427	−0·06536	0·01829	0·08816	0·95311
0·3	0·52440	−0·34177	−0·02646	−0·51857	−0·05063	0·00996	0·03670	0·23275
0·2	0·84162	−0·13750	−0·03572	−0·67018	−0·02037	−0·00523	−0·05447	−1·01408
0·1	1·28155	0·30282	−0·03222	−0·48849	0·04486	−0·02903	−0·18408	−2·63128
0·05	1·64485	0·80400	−0·00897	0·15025	0·11911	−0·04130	−0·22040	−2·69255
0·01	2·32635	2·07979	0·10391	3·01071	0·30812	−0·00128	0·22152	5·70110
0·005	2·57583	2·65632	0·17339	4·73369	0·39353	0·05037	0·67292	13·52149
0·001	3·09022	4·03026	0·37480	9·68195	0·59708	0·25767	2·38033	42·26471

4. Tables 2 and 3 give the percentage points and the probability integral for the χ'^2 distribution as calculated by the first approximation and by the closer approximation. For sake of comparison we have quoted the corresponding values obtained by Patnaik (1949) using a χ^2 distribution. It will be seen that even for small degrees of freedom and moderate values of λ the expansion agrees very well with the exact value. In general, the results using the first approximation are of about the same accuracy as those obtained by Patnaik's central χ^2 fit, which is to be expected in view of the properties of the Wilson-Hilferty transformation. Patnaik also considered an approximation based on applying the Fisher-Cornish technique directly to χ'^2; our results have been more successful because we have applied it to $(\chi'^2)^{\frac{1}{3}}$, i.e. to a variable more nearly normally distributed to start with.

Table 2. *Percentage points of the χ'^2-distribution*

f	λ	Upper 5% point				Lower 5% point			
		χ^2 fit	First approx.	Closer approx.	Exact	χ^2 fit	First approx.	Closer approx.	Exact
2	1	8·63	8·56	8·38	8·642	0·20	0·17	0·171	0·168
	4	14·72	14·66	14·62	14·641	0·94	0·89	0·643	0·646
	16	33·35	33·32	33·08	33·054	6·89	6·87	6·320	6·322
	25	45·66	45·64	45·33	45·308	12·68	12·67	12·077	12·080
4	1	11·72	11·67	11·67	11·707	0·93	0·91	0·908	0·909
	4	17·38	17·34	17·27	17·309	1·95	1·93	1·766	1·765
	16	35·69	35·66	35·44	35·427	8·36	8·35	7·885	7·884
	25	47·94	47·91	47·62	47·613	14·26	14·26	13·732	13·733
7	1	16·01	15·98	15·99	16·004	2·51	2·49	2·494	2·494
	4	21·28	21·25	21·21	21·228	3·78	3·76	3·665	3·664
	16	39·16	39·16	38·96	38·970	10·64	10·63	10·256	10·257
	25	51·34	51·33	51·06	51·061	16·68	16·67	16·227	16·226

Table 3. *Probability integral* $\int_0^{\chi'^2} p(\chi'^2)\, d\chi'^2$

f	λ	χ'^2	χ^2 fit	First approx.	Closer approx.	Exact
4	4	10·00	0·7191	0·7198	0·7123	0·7118
4	4	24·00	0·9913	0·9911	0·9925	0·9925
7	16	24·00	0·5947	0·5947	0·5894	0·5898
7	16	38·97	0·9482	0·9484	0·9500	0·9500
16	8	20·00	0·3380	0·3374	0·3368	0·3369
16	8	40·00	0·9626	0·9626	0·9632	0·9632

My thanks are due to Dr F. N. David whose suggestions helped considerably in the presentation of this paper.

REFERENCES

Cornish, E. A. & Fisher, R. A. (1937). *Rev. Inst. Int. Statist.* **5**, 307.
Fisher, R. A. (1928). *Proc. Roy. Soc.* A, **121**, 654.
Fix, E. (1949). *Univ. Calif. Publ. Statist.* **1**, 15.
Garwood, F. (1934). Unpublished Ph.D. Thesis for University of London.
Kendall, M. G. (1947). *The Advanced Theory of Statistics*, **1**. London: Chas. Griffin and Co. Ltd.
Patnaik, P. B. (1949). *Biometrika*, **36**, 204.

Moments of the mean deviation

By A. R. KAMAT
Fergusson College, Poona, India

In an earlier paper (Kamat, 1953) I have derived formulae for bivariate and trivariate absolute moments of correlated normal variates. As a direct application of these results, one may derive the moments of the mean deviation in samples of size n from a normal population. Thus if

$$m = \sum_{i=1}^{n} |x_i - \bar{x}|/n,$$

and we write $z_i = x_i - \bar{x}$, then the z_i are normal variates having zero expectation, variance $\sigma^2(n-1)/n$ and correlations $\rho = \rho(z_i, z_j) = -1/(n-1)$ $(i \neq j)$.

Using the exact results of § 2·6 of my earlier paper and, for $\mathscr{E}(|z_i z_j z_k z_l|)$, the expansion in a power series given in § 3, the first four moments of m can be derived, the last involving an expansion in inverse powers of $n-1$. The values of the first two moments are of course well known, while Geary's work (1936), led to expansions for the third and fourth moment quoted by Pearson (1945).*

Prof. Pearson and Dr Hartley have, however, suggested that I should put the following results on record as they were used in an investigation into the accuracy of values for the beta coefficients for low n, quoted in Table 20 of their revised (1954) *Biometrika Tables for Statisticians*.

(*a*) *Third moment of m*

This can be expressed *exactly* as follows:

$$\frac{\mu_3}{\sigma^3} = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left\{ \sqrt{\left(\frac{2}{\pi}\right)} \left(\frac{4-n}{n^2(n-1)}\right) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[\frac{n-2}{n} \left(\frac{n-3}{n-1}\right)^{\frac{1}{2}} + \frac{3(n-2)^2}{n^2(n-1)} \sin^{-1} \frac{1}{n-2} \right.\right.$$
$$\left.\left. + 2 - \frac{3(n-1)}{n} \left(1 - \frac{1}{(n-1)^2}\right)^{\frac{1}{2}} - \frac{3}{n} \sin^{-1} \frac{1}{n-1} \right] \right\}. \quad (1)$$

(*b*) *Fourth moment of m*

$$\frac{\mu_4}{\sigma^4} = \left(\frac{n-1}{n}\right)^2 \left\{ \frac{3(n^2-n+2)}{n^3(n-1)} + \frac{2}{\pi} \left[\frac{6n^3-16n^2+26n-24}{n^3(n-1)} \left(1 - \frac{1}{(n-1)^2}\right)^{\frac{1}{2}} + \frac{6(n^2-3n+4)}{n^3(n-1)} \sin^{-1} \frac{1}{n-1} \right.\right.$$
$$\left. - \frac{6n^2-10n+16}{n^2(n-1)} \right] + \left(\frac{2}{\pi}\right)^2 \left[\frac{6(n-1)}{n} \left(1 - \frac{1}{(n-1)^2}\right)^{\frac{1}{2}} - 3 - \frac{4(n-2)}{n} \left(\frac{n-3}{n-1}\right)^{\frac{1}{2}} \right.$$
$$\left.\left. + \frac{6}{n} \sin^{-1} \frac{1}{n-1} - \frac{12(n-2)^2}{n^2(n-1)} \sin^{-1} \frac{1}{n-2} + \frac{(n-1)(n-2)(n-3)}{n^3} A \right] \right\},$$

where

$$A = 1 + 3(n-1)^{-2} - 4(n-1)^{-3} + (n-1)^{-4} + 4(n-1)^{-5} + \tfrac{31}{5}(n-1)^{-6} + \tfrac{48}{5}(n-1)^{-7} + \tfrac{382}{25}(n-1)^{-8} + \ldots. \quad (2)$$

The series involved in (2), while different in form from that given by Geary, which he obtained by direct integration, is asymptotically equivalent.

Table 1 gives numerical values of the β_1 and β_2 coefficients of the mean deviation for sample sizes $n = 3$, 4, 5 and 6 calculated:

(*a*) From equations (1) and (2) above.

(*b*) From the Geary expansions (3) and (4) of *Biometrika*, **33**, 252 (1945) with Godwin's (1948) correction.

(*c*) By numerical quadrature of Godwin & Hartley's (1945) table of the probability integral of m.

Our values for β_1 (i.e. those given under (*a*)) may be taken as correct values since they are obtained from the exact formula for μ_3 given above. The values obtained from Geary's expansion for μ_3 agree with them for $n > 4$. In the case of β_2, the value for $n = 3$ given by us is exact, since the formula for μ_4 in this case is free of the expansion for $\mathscr{E}(|z_i z_j z_k z_l|)$. For $n = 4$, the value of $\beta_2 = 3{\cdot}252$ obtained from Geary's expansion coincides with that calculated from the exact formula obtained from the distribution given

* A correction to the last term in the expression for $\lambda_4 = \mu_4 - 3\mu_2^2$ was noted by Godwin (1948).

by Fisher (1920).* We may conclude that in the case of $n = 5$, also, the value of β_2 obtained from Geary's expansion is to be preferred to that obtained by using the expression for μ_4 given in (2). From $n = 6$ onwards the two formulae give identical values for β_2. Geary's expansion for λ_4 is given up to the term in $(n-1)^{-6}$ and it seems at first sight paradoxical that it should provide a better approximation than the formula (2) for μ_4 which contains a series expanded up to $(n-1)^{-3}$. The result may, however, be due to the expansion of all other terms in $\lambda_4 = \mu_4 - 3\mu_2^2$ in inverse powers of $(n-1)$ which probably corrects the error involved in the use of the series for $\mathscr{E}(|z_i z_j z_k z_l|)$.

Table 1. *Comparison of β_1, β_2 obtained by different methods described above*†

n	3	4	5	6
β_1 (*a*) formula (1)	0·417	0·298	0·230	0·187
(*b*) Geary's expansion	—	0·299	0·230	0·187
(*c*) quadrature	0·418	0·298	0·229	0·186
β_2 (*a*) formula (2)	3·286	3·243	3·195	3·161
(*b*) Geary's expansion	—	3·252	3·197	3·161
(*c*) quadrature	3·286	3·251	3·196	3·159

REFERENCES

Cadwell, J. H. (1953). *Biometrika*, **40**, 336.
Fisher, R. A. (1920). *Mon. Not. R. Astr. Soc.* **80**, no. 8, 758.
Geary, R. C. (1936). *Biometrika*, **28**, 295.
Godwin, H. J. (1948). *Biometrika*, **35**, 304.
Godwin, H. J. & Hartley, H. O. (1945). *Biometrika*, **33**, 254.
Kamat, A. R. (1953). *Biometrika*, **40**, 20.
Pearson, E. S. (1945). *Biometrika*, **33**, 252 (Editorial Note).
Pearson, E. S. & Hartley, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge: The University Press for the Biometrika Trust.

On the relation between estimating efficiency and the power of tests

By R. M. SUNDRUM

University of Rangoon and Institute of Statistics, University of North Carolina‡

It appears to be generally assumed that a statistic which has a high efficiency in estimating an unknown parameter also gives a powerful test of hypotheses about that parameter. For example, a common criterion for comparing distribution-free tests with corresponding distributional tests and with other distribution-free tests is the asymptotic relative efficiency first proposed by Pitman (1948). Recently, Stuart (1954) has shown that this is equivalent to using the estimating efficiency of these statistics to compare their performance when used in tests of significance. In fact, it was by this argument that this measure of efficiency was earlier arrived at by Hotelling & Pabst (1936) in their study of the Spearman rank correlation coefficient and by Cochran (1937) in his study of two tests of the mean and of the correlation coefficient of normally distributed variables. In this note, we derive a condition for the validity of this assumption in the case when the statistics have normal distributions, a case for which the idea of estimating efficiency has most relevance.

* For $n = 4$ the expression for μ_3 derived by Fisher is the same as given in our equation (1) above, but, as pointed out by Cadwell (1953, p. 342), a correction is needed in Fisher's expression for the fourth moment; the necessary alteration gives a correct value of $\beta_2 = 3{\cdot}252$.

† I am indebted to Miss May and Mr Ablewhite of the Department of Statistics, University College, London, for some of the computations in this connexion.

‡ This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

Let t_1 and t_2 be two statistics which are normally distributed, unbiased estimators of a population parameter θ with variances $\sigma_1^2(\theta_0)$ and $\sigma_2^2(\theta_0)$ respectively under the null hypothesis H_0: $\theta = \theta_0$; and $\sigma_1^2(\theta_1)$ and $\sigma_2^2(\theta_1)$ respectively under the alternative hypothesis H_1: $\theta = \theta_1 > \theta_0$. Let $\sigma_1^2(\theta_0) \leqslant \sigma_2^2(\theta_0)$, $\sigma_1^2(\theta_1) \leqslant \sigma_2^2(\theta_1)$, the inequality holding at least once, so that t_1 may be called the more efficient estimator.

If λ_α is defined by $$\Phi(\lambda_\alpha) = \frac{1}{\sqrt{(2\pi)}} \int_{\lambda_\alpha}^{\infty} e^{-\frac{1}{2}y^2} dy = \alpha,$$

the one-sided critical region of size α based on t_1 is given by

$$t_1 > \theta_0 + \lambda_\alpha \sigma_1(\theta_0), \tag{1}$$

and for t_2 by

$$t_2 > \theta_0 + \lambda_\alpha \sigma_2(\theta_0). \tag{2}$$

The power of the critical region (1) is then

$$P_1 = \Phi\left\{\frac{\theta_0 + \lambda_\alpha \sigma_1(\theta_0) - \theta_1}{\sigma_1(\theta_1)}\right\} \tag{3}$$

and of the critical region (2) is

$$P_2 = \Phi\left\{\frac{\theta_0 + \lambda_\alpha \sigma_2(\theta_0) - \theta_1}{\sigma_2(\theta_1)}\right\}. \tag{4}$$

As $\Phi(x)$ is a monotonically decreasing function of x, the test based on t_1 (the more efficient estimator) is more powerful if

$$\frac{\theta_0 - \theta_1 + \lambda_\alpha \sigma_1(\theta_0)}{\sigma_1(\theta_1)} < \frac{\theta_0 - \theta_1 + \lambda_\alpha \sigma_2(\theta_0)}{\sigma_2(\theta_1)}, \tag{5}$$

i.e. if

$$\theta_1 - \theta_0 > \lambda_\alpha \left\{\frac{\sigma_1(\theta_0)\,\sigma_2(\theta_1) - \sigma_2(\theta_0)\,\sigma_1(\theta_1)}{\sigma_2(\theta_1) - \sigma_1(\theta_1)}\right\} = \lambda_\alpha \left(\frac{V_1 - V_0}{V_1 - 1}\right) \sigma_1(\theta_0),$$

where

$$V_i = \frac{\sigma_2(\theta_i)}{\sigma_1(\theta_i)} \quad (i = 1, 2).$$

If $V_0 \geqslant V_1$, then the right-hand side of (5) is less than or equal to 0 and the inequality is satisfied. But if $V_1 > V_0 \geqslant 1$, so that the relative efficiency of t_1 is greater under the alternative than under the null hypothesis, then $\left(\frac{V_1 - V_0}{V_1 - 1}\right) > 0$. As α, the size of the critical region is always taken less than $\frac{1}{2}$ in practical applications, λ_α is always positive and can be chosen so large that

$$\lambda_\alpha \left(\frac{V_1 - V_0}{V_1 - 1}\right) \sigma_1(\theta_0) > \theta_1 - \theta_0$$

and the inequality (5) is reversed.

However, this possibility is not very important in practice. For if λ_α is chosen so as to reverse the inequality (5), the power of the test will be reduced very much. In fact under the special assumptions of the above argument, the situation in which the more efficient estimator gives a less powerful test cannot arise, if the level of significance is chosen so as to make the power of that test greater than $\frac{1}{2}$. This can be seen as follows:

$$P_1 = \Phi\left(\frac{\theta_0 + \lambda_\alpha \sigma_1(\theta_0) - \theta_1}{\sigma_1(\theta_1)}\right) > \tfrac{1}{2} = \Phi(0)$$

if

$$\theta_0 - \theta_1 + \lambda_\alpha \sigma_1(\theta) < 0,$$

i.e. if

$$(\theta_1 - \theta_0) > \lambda_\alpha \sigma_1(\theta_0). \tag{6}$$

From the condition $V_1 > V_0 \geqslant 1$, we have

$$0 < \frac{V_1 - V_0}{V_1 - 1} \leqslant 1. \tag{7}$$

From (6) and (7) the inequality (5) follows.

The argument of this note helps to explain the fact mentioned by Fisher (1950, pp. 314–15) that good tests may be based on inefficient estimators. He gives an example from genetics concerning a test for linkage in inheritance of two factors. Given the frequencies of four combinations in a sample as a, b, c and d with

$$a + b + c + d = n \tag{8}$$

and the corresponding probabilities of occurrence as

$$\tfrac{1}{4}(2+\theta), \quad \tfrac{1}{4}(1-\theta), \quad \tfrac{1}{4}(1-\theta), \quad \tfrac{1}{4}\theta,$$

the problem is one of estimating θ, when $\sqrt{\theta}$ is the recombination ratio. The maximum likelihood method gives a statistic t_1 as the positive solution of the equation

$$n\theta^2 - (a - 2b - 2c - d)\theta - 2d = 0 \tag{9}$$

with a sampling variance

$$\sigma^2(t_1) = \frac{2\theta(1-\theta)(2+\theta)}{(1+2\theta)n}. \tag{10}$$

Another statistic t_2 may be defined by

$$t_2 = \frac{a - b - c + 5d}{2n} \tag{11}$$

with expectation θ and sampling variance

$$\sigma^2(t_2) = \frac{1 + 6\theta - 4\theta^2}{4n}. \tag{12}$$

It is easily verified that $\sigma^2(t_1) \leqslant \sigma^2(t_2)$.

Now consider a test of H_0: $\theta = \frac{1}{4}$ (no linkage) against H_1: $\theta = \theta_1 > \frac{1}{4}$. We find $V_0 = 1$ so that, assuming n large enough for a normal approximation to hold satisfactorily for the distributions of t_1 and t_2, and for the bias of the maximum likelihood estimator to be negligible, the inequality (5) is reversed for all θ_1 satisfying

$$(\theta_1 - \tfrac{1}{4}) < \frac{3\lambda_\alpha}{4\sqrt{n}}. \tag{13}$$

REFERENCES

Cochran, W. G. (1937). *J.R. Statist. Soc.* **100**, 69.

Fisher, R. A. (1950). *Statistical Methods for Research Workers*, 11th ed. Edinburgh: Oliver and Boyd.

Hotelling, H. & Pabst, M. (1936). *Ann. Math. Statist.* **7**, 29.

Pitman, E. J. G. (1948). *Lectures on Non-Parametric Inference*. University of North Carolina.

Stuart, Alan (1954). *J. Amer. Statist. Ass.* **49**, 147.

The interpretation of negative components of variance

By J. A. NELDER

National Vegetable Research Station

1. Introduction

It not infrequently happens in split-plot experiments that the error mean square for main plots is less than the error mean square for subplots. Under the normal model the expected value of the error mean square for main plots is $\sigma^2 + r\sigma'^2$, and for subplots, σ^2, where σ^2 is the subplot component of error, σ'^2 is the main plot component of error, and r is the number of subplots in each main plot. Thus, if this model holds, the main plot error in an experiment can only be less than the subplot error by sampling deviations. This fact seems to have led expositors of the analysis of variance to recommend that when the main-plot error is less than the subplot error in an analysis, σ'^2 should be taken as zero, and the two errors should be bulked. On the randomization model, however, there is no reason why the main-plot error should not be less than the subplot error in expected value as well as in any particular experimental analysis. Furthermore, an unbiased estimate of variance of treatment differences is obtained by using the usual formulae as though σ'^2 were negative. This note will present a modified form of the normal model, originally due to F. J. Anscombe (unpublished lecture notes), which has a closer affinity to the randomization model than the customary normal model, and in which negative components of variance can appear and be given a satisfactory interpretation. We consider first the randomized blocks type of experiment. The argument was outlined by Anscombe (1948) in a contribution to the discussion of a paper by Champernowne.

2. Randomized blocks

Consider a set of fixed numbers x_{ij} ($i = 1, \ldots, r$; $j = 1, \ldots, t$), where x_{ij} is the datum from the jth plot in the ith block. From these numbers form a set of random variables y_{ij} as follows: choose a block at random and reorder its members at random to give $y_{11}, y_{12}, \ldots, y_{1t}$; repeat the procedure with one of the remaining

blocks chosen at random to give $y_{21}, \ldots, y_{2t}$, and so on. The distribution of the y's has the following first- and second-order moments [summation is over all combinations of suffixes except those indicated]:

$$\mu_1'(y) = \frac{1}{n}\Sigma x_{ij} = \mu, \quad n = rt,$$

$$\text{var}\,(y) = \frac{1}{n}\Sigma x_{ij}^2 - \mu^2 = \sigma^2,$$

$$\text{cov}\,(y_{ij}, y_{il}) = \frac{1}{n(t-1)} \sum_{j \neq l} x_{ij}x_{il} - \mu^2 = \rho_1\sigma^2 \quad (j \neq l),$$

$$\text{cov}\,(y_{ij}, y_{kl}) = \frac{1}{nt(r-1)} \sum_{i \neq k} x_{ij}x_{kl} - \mu^2 = \rho_2\sigma^2 \quad (i \neq k).$$

Suppose now that the distribution of the y's can be approximated to by the multivariate normal distribution with the same first- and second-order moments; then the cumulant generating function of the distribution will be

$$\Psi(t_{ij}) = \mu\Sigma t_{ij} + \frac{\sigma^2}{2}[\Sigma t_{ij}^2 + 2\rho_1 \sum_{j \neq l} t_{ij}t_{il} + 2\rho_2 \sum_{i \neq k} t_{ij}t_{kl}]$$

$$\equiv \frac{\sigma^2}{2}(1-\rho_1)\Sigma t_{ij}^2 + \frac{\sigma^2}{2}(\rho_1 - \rho_2)\sum_i (\Sigma_j t_{ij})^2 + \left[\mu\Sigma t_{ij} + \frac{\sigma^2}{2}\rho_2(\Sigma t_{ij})^2\right].$$

Thus we can write

$$y_{ij} = m + b_i + e_{ij},$$

where

$$E(m) = \mu, \quad \text{var}\,(m) = \rho_2\sigma^2,$$

$$E(b_i) = 0, \quad \text{var}\,(b_i) = (\rho_1 - \rho_2)\sigma^2,$$

$$E(e_{ij}) = 0, \quad \text{var}\,(e_{ij}) = (1-\rho_1)\sigma^2,$$

and the m, b_i and e_{ij} are all independent normal variables. The randomization distribution imposes restrictions on ρ_1 and ρ_2. We can, without loss of generality, put $\mu = 0$; then

$$n\sigma^2[1 + (t-1)\rho_1] = \sum_{j \neq l} x_{ij}x_{il} + \Sigma x_{ij}^2 = \sum_i (\Sigma_j x_{ij})^2 \geqslant 0,$$

$$n\sigma^2[1 + (t-1)\rho_1 + t(r-1)\rho_2] = \sum_i (\Sigma_j x_{ij})^2 + \sum_{i \neq k} x_{ij}x_{kl} - (\Sigma x_{ij})^2$$

$$= (\Sigma x_{ij})^2 - (\Sigma x_{ij})^2 = 0.$$

It follows from this that $\rho_2 \leqslant 0$. Thus the variance of m is negative, and the variance of b_i may be negative, for it is possible to have $\rho_1 < \rho_2$. The model $y_{ij} = m + b_i + e_{ij}$, therefore, is to be understood as a shorthand way of expressing the multivariate distribution of the y's. It is not difficult to show that the complete distribution is a valid one, i.e. that the variance matrix is positive-semi-definite. The variance matrix may be written

$$\mathbf{V} = \sigma^2 \begin{bmatrix} \mathbf{A} & \mathbf{B} & \cdots & \cdots & \cdots & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & & & \mathbf{B} \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \mathbf{B} \\ \mathbf{B} & & & & \mathbf{B} & \mathbf{A} \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & \rho_1 & \cdots & \cdots & \cdots & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & & \rho_1 \\ \vdots & & 1 & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \rho_1 \\ \rho_1 & & & & \rho_1 & 1 \end{bmatrix} = (1-\rho_1)\mathbf{I} + \rho_1\mathbf{J},$$

$\mathbf{I}$ being the unit matrix and $\mathbf{J}$ the matrix with all elements unity, and $\mathbf{B} = \rho_2\mathbf{J}$. [$\mathbf{A}$ and $\mathbf{B}$ are both square matrices of order t.] Consider now $|\mathbf{V} - \lambda\sigma^2\mathbf{I}|$; by adding all rows together we find that

$$[1 + (t-1)\rho_1 + t(r-1)\rho_2 - \lambda]$$

is a factor. Subtracting any block of t rows from any other we get $[1+(t-1)\rho_1-r\rho_2-\lambda]$ as a factor and this may be obtained from $r-1$ linearly independent operations. Finally, subtracting any row within a block from any other we have $(1-\rho_1-\lambda)$ as a factor and this may be obtained from $r(t-1)$ linearly independent operations. Hence

$$|\mathbf{V}-\lambda\sigma^2\mathbf{I}| = \sigma^{2n}[1+(t-1)\rho_1+t(r-1)\rho_2-\lambda][1+(t-1)\rho_1-t\rho_2-\lambda]^{r-1}[1-\rho_1-\lambda]^{r(t-1)}.$$

Under randomization the first latent root is zero corresponding to the fact that $\text{var}(\Sigma y_{ij}) = 0$, the second repeated latent root equals $-tr\rho_2 \geqslant 0$ and the third is $1-\rho_1 \geqslant 0$. All the latent roots are thus non-negative and the distribution is a valid one, though degenerate in that the rank of $\mathbf{V}$ cannot exceed $n-1$. The three distinct latent roots multiplied by σ^2 give the expected values of the variance of the mean, the between-blocks mean square, and the within-blocks mean square respectively, while the number of repetitions of each root gives the degrees of freedom for each sum of squares. The expected mean square for blocks will be less than the expected mean square for within-blocks if

$$1+(t-1)\rho_1-t\rho_2 < 1-\rho_1, \quad \text{i.e. if} \quad \rho_1 < \rho_2.$$

The possible values for ρ_1 and ρ_2 are shown in Fig. 1, together with the region where the block 'component of variance' is negative.

3. Split-plot experiments

The situation in split-plot experiments is slightly more complicated, but not essentially different. Let x_{ijk} $(i = 1, \ldots, r;\ j = 1, \ldots, s;\ k = 1, \ldots, t)$ be the quantities from which the randomization distribution is supposed to be generated, where x_{ijk} refers to the kth subplot in the jth main plot in the ith block. Let y_{ijk} be the randomized variates produced from the randomization procedure. There are three different correlation coefficients between pairs of y's according as they lie in the same main plot, in the same block but not in the same main plot, or in different blocks. Denote these by ρ_1, ρ_2 and ρ_3 respectively. The first- and second-order moments of the distribution of the y's are given by

$$\mu_1'(y) = \frac{1}{n}\Sigma x_{ijk} = \mu, \quad n = rst,$$

$$\text{var}(y) = \frac{1}{n}\Sigma x_{ijk}^2 - \mu^2 = \sigma^2,$$

$$\rho_1\sigma^2 = \frac{1}{n(t-1)} \sum_{k \neq k'} x_{ijk}x_{ijk'} - \mu^2,$$

$$\rho_2\sigma^2 = \frac{1}{nt(s-1)} \sum_{j \neq j'} x_{ijk}x_{ij'k'} - \mu^2,$$

$$\rho_3\sigma^2 = \frac{1}{nst(r-1)} \sum_{i \neq i'} x_{ijk}x_{i'j'k'} - \mu^2.$$

By an obvious extension of the randomized block argument we have

$$1+(t-1)\rho_1 \geqslant 0,$$

$$1+(t-1)\rho_1+t(s-1)\rho_2 \geqslant 0,$$

and

$$1+(t-1)\rho_1+t(s-1)\rho_2+ts(r-1)\rho_3 = 0.$$

The scope of possible values for ρ_1 and ρ_2 is now considerably widened, as is shown in Fig. 2. Again the main-plot component of variance will be negative if $\rho_1 < \rho_2$.

4. Comparison of the models

The modified normal model presented above bears a closer relation to the randomization model than the orthodox normal model, in that its first- and second-order moments agree exactly with those of the randomization distribution. However, the analogy between the modified normal model and the randomization model must not be carried too far; for example, the between-blocks and within-blocks sums of squares are constants in the randomization model, but are not so in the modified normal model. The two normal models are identical in many respects, except that the expression of a datum as the sum of several independent random variables is a formal one in the modified scheme, in that some of the variables may have negative variances. Many of the deductions from the orthodox normal model remain true with the modified one. For example, the F-distribution still holds for the treatments-error mean square ratio on the null hypothesis; this is because any two contrasts contributing to the within-blocks sum of

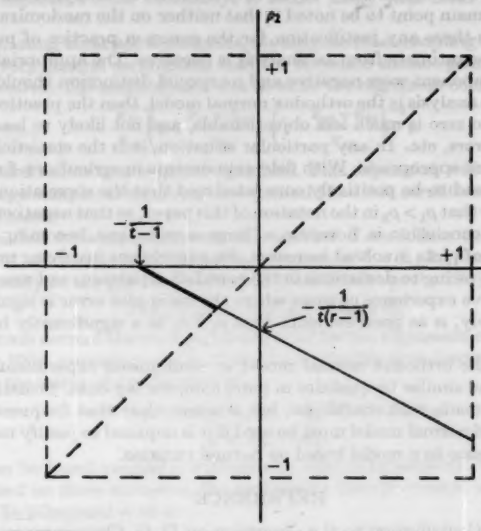


Fig. 1. Possible pairs of values for ρ_1 and ρ_2 in a randomized block experiment lie on the solid diagonal line. On the thickened part, the block component of variance is negative.

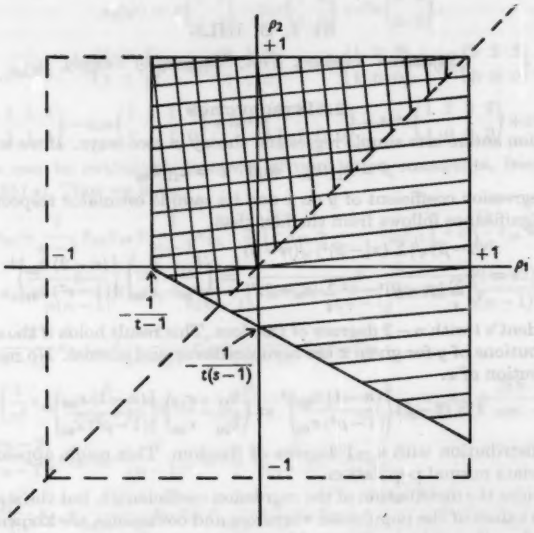


Fig. 2. Permissible values of ρ_1 and ρ_2 in split-plot experiments. Single shading: permissible area, main-plot component of variance positive. Double shading: permissible area, main-plot component of variance negative.

squares, which are independent and have equal variances on the orthodox normal model, remain so on the modified one. The main point to be noted is that neither on the randomization model nor on the modified normal model is there any justification for the common practice of putting equal to zero a variance component whose estimate from an analysis is negative. The appropriate formulae should be applied as though the component were negative and no special distinction should be made.

If the model used in the analysis is the orthodox normal model, then the practice of equating negative components of variance to zero is much less objectionable, and not likely to lead to much bias in the estimation of standard errors, etc. In any particular situation, it is the statistician's responsibility to decide which model is more appropriate. With field experiments in agriculture, for example, it is known that neighbouring plots tend to be positively correlated and that the correlation usually falls off with distance; this would imply that $\rho_1 > \rho_2$ in the notation of this paper, so that negative variance components would not occur. Such a conclusion is, however, a 'large-sample' one, becoming more and more likely to be true as the number of plots involved increases. An experiment involving only a small number of plots may well give $\rho_1 < \rho_2$ owing to deviations in the correlation pattern, and most statisticians dealing with such experiments have experience of cases where the main-plot error is significantly less than the split-plot error. This, surely, is as good evidence that $\rho_1 < \rho_2$ as a significantly higher main-plot error is that $\rho_1 > \rho_2$.

In general the use of the orthodox normal model in randomized experiments implies an *a priori* assumption that $\rho_1 > \rho_2$ and similar inequalities in more complex lay-outs. Situations may occur where such assumptions can be made with confidence, but it seems clear that frequently they cannot, with the result that the modified normal model must be used if it is required to justify the analysis of randomized experiments by reference to a model based on normal variates.

REFERENCE

ANSCOMBE, F. J. (1948). Contribution to the discussion on D. G. Champernowne's 'Sampling theory applied to autoregressive sequences'. *J.R. Statist. Soc.* B, **10**, 239.

The distribution of the regression coefficient in samples from a non-normal population

BY I. D. HILL

Statistical Advisory Unit, Ministry of Supply

1. INTRODUCTION

Student's t-distribution enters into simple regression theory in two ways. If we write

$$\beta = \kappa_{11}/\kappa_{20} \quad \text{and} \quad b = k_{11}/k_{20}$$

for the population regression coefficient of y on x and its sample estimator respectively, then the commonly used test of significance follows from the fact that

$$t = \frac{(b-\beta)\sqrt{\{\sum_i (x_i-\bar{x})^2\}}\sqrt{(n-2)}}{\sqrt{\{\sum_i (y_i-\bar{y})^2 - b^2 \sum_i (x_i-\bar{x})^2\}}} = \left(\frac{k_{11}}{k_{20}} - \frac{\kappa_{11}}{\kappa_{20}}\right)\left\{\frac{(n-2)\,k_{20}}{(1-r^2)\,k_{02}}\right\}^{\frac{1}{2}} \tag{1}$$

is distributed as Student's t with $n-2$ degrees of freedom. This result holds if the regression of y on x is linear and the distributions of y for given x are homoscedastic and normal. No restriction need here be placed on the distribution of x.

Secondly,
$$t = (b-\beta)\left\{\frac{(n-1)\,\kappa_{20}}{(1-\rho^2)\,\kappa_{02}}\right\}^{\frac{1}{2}} = \left(\frac{k_{11}}{k_{20}} - \frac{\kappa_{11}}{\kappa_{20}}\right)\left\{\frac{(n-1)\,\kappa_{20}}{(1-\rho^2)\,\kappa_{02}}\right\}^{\frac{1}{2}} \tag{2}$$

follows Student's t-distribution with $n-1$ degrees of freedom. This result appears to be true only in samples from a bivariate normal population.

The result (2) provides the distribution of the regression coefficient, b, but the statistic cannot be used in practice unless the values of the population variances and covariance are known. On the other hand, without this knowledge, the statistic (1) can be used to test whether an observed value of b differs significantly from zero or from a specified β, and to obtain confidence limits for β given b.

In this paper it will be considered how far departure from normality affects the distribution of b, that is to say how far it modifies the significance levels that result from (2). The question of the modification to levels derived from (1) will not be considered directly, although in so far as tests based on (2)

are not sensitive to departure from normality, it seems likely that tests based on (1) will also be insensitive.

M. B. Cook (1951*b*), using bivariate k-statistics, investigated the sampling moments of the regression coefficient under the very general assumption that the bivariate surface generating the samples had as many cumulants as desired. The method used was to write the regression coefficient, k_{11}/k_{20}, in the form

$$\frac{\kappa_{11}}{\kappa_{20}}\left(1+\frac{k_{11}-\kappa_{11}}{\kappa_{11}}\right)\left(1+\frac{k_{20}-\kappa_{20}}{\kappa_{20}}\right)^{-1},$$

and to expand the third factor as a negative binomial. This expansion will be valid only if $k_{20}<2\kappa_{20}$, but for reasonably large sample sizes, it can be expected that this condition will be satisfied in all but a small proportion of cases.

Difficulties such as this with expansions can be avoided if we follow the technique developed by David & Johnson (1951) and, instead of finding the moments of $b=k_{11}/k_{20}$, examine the manner in which departure from normality in the parent distribution modifies the probability,

$$P\{k_{11}/k_{20}>c\}=P\{k_{11}-ck_{20}>0\}, \tag{3}$$

where c is an appropriate normal theory significance level for the regression coefficient derived from (2).

If $t_{\nu,\alpha}$ is the upper 100α percentage point of Student's distribution having ν degrees of freedom, it follows from (2) that the value of c to be used is

$$c=\{\kappa_{11}+t_{n-1,\alpha}(\kappa_{20}\kappa_{02}-\kappa_{11}^2)^{\frac{1}{2}}/(n-1)^{\frac{1}{2}}\}/\kappa_{20}. \tag{4}$$

Writing

$$\psi=k_{11}-ck_{20},$$

the moments of ψ can be found precisely, without any infinite expansions. Then, using an appropriate graduating curve based on these moments, the probability that ψ exceeds zero can be estimated. $P\{\psi>0\}$ can then be compared with α.

2. Derivation of the moments of $\psi=k_{11}-ck_{20}$

We have

$$\mathscr{E}(\psi)=\kappa_{11}-c\kappa_{20} \tag{5}$$

and

$$(\psi-\mathscr{E}(\psi))^2=(k_{11}-\kappa_{11})^2-2c(k_{11}-\kappa_{11})(k_{20}-\kappa_{20})+c^2(k_{20}-\kappa_{20})^2,$$

or

$$\mu_2(\psi)=\kappa\begin{pmatrix}1&1\\1&1\end{pmatrix}-2c\kappa\begin{pmatrix}1&2\\1&0\end{pmatrix}+c^2\kappa\begin{pmatrix}2&2\\0&0\end{pmatrix}.$$

Similarly

$$\mu_3(\psi)=\kappa\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}-3c\kappa\begin{pmatrix}1&1&2\\1&1&0\end{pmatrix}+3c^2\kappa\begin{pmatrix}1&2&2\\1&0&0\end{pmatrix}-c^3\kappa\begin{pmatrix}2&2&2\\0&0&0\end{pmatrix},$$

$$\kappa_4(\psi)=\kappa\begin{pmatrix}1&1&1&1\\1&1&1&1\end{pmatrix}-4c\kappa\begin{pmatrix}1&1&1&2\\1&1&1&0\end{pmatrix}+6c^2\kappa\begin{pmatrix}1&1&2&2\\1&1&0&0\end{pmatrix}-4c^3\kappa\begin{pmatrix}1&2&2&2\\1&0&0&0\end{pmatrix}+c^4\kappa\begin{pmatrix}2&2&2&2\\0&0&0&0\end{pmatrix}.$$

These expressions can be evaluated, in terms of population cumulants, from tables of bivariate k-statistics (Cook, 1951*a*). Thus we have

$$\mu_2(\psi)=\frac{1}{n}\kappa_{22}+\frac{1}{n-1}\kappa_{20}\kappa_{02}+\frac{1}{n-1}\kappa_{11}^2-2c\left(\frac{1}{n}\kappa_{31}+\frac{2}{n-1}\kappa_{20}\kappa_{11}\right)+c^2\left(\frac{1}{n}\kappa_{40}+\frac{2}{n-1}\kappa_{20}^2\right), \tag{6}$$

$$\begin{aligned}\mu_3(\psi)=&\frac{1}{n^2}\kappa_{33}+\frac{6}{n(n-1)}\kappa_{22}\kappa_{11}+\frac{3}{n(n-1)}\kappa_{20}\kappa_{13}+\frac{3}{n(n-1)}\kappa_{31}\kappa_{02}+\frac{3(n-2)}{n(n-1)^2}\kappa_{21}\kappa_{12}\\&+\frac{n-2}{n(n-1)^2}\kappa_{30}\kappa_{03}+\frac{2}{(n-1)^2}\kappa_{11}^3+\frac{6}{(n-1)^2}\kappa_{11}\kappa_{20}\kappa_{02}\\&-3c\left(\frac{1}{n^2}\kappa_{42}+\frac{5}{n(n-1)}\kappa_{22}\kappa_{20}+\frac{6}{n(n-1)}\kappa_{31}\kappa_{11}+\frac{1}{n(n-1)}\kappa_{40}\kappa_{02}+\frac{2(n-2)}{n(n-1)^2}\kappa_{21}^2\right.\\&\left.+\frac{2(n-2)}{n(n-1)^2}\kappa_{30}\kappa_{12}+\frac{6}{(n-1)^2}\kappa_{20}\kappa_{11}^2+\frac{2}{(n-1)^2}\kappa_{20}^2\kappa_{02}\right)\\&+3c^2\left(\frac{1}{n^2}\kappa_{51}+\frac{8}{n(n-1)}\kappa_{31}\kappa_{20}+\frac{4}{n(n-1)}\kappa_{40}\kappa_{11}+\frac{4(n-2)}{n(n-1)^2}\kappa_{30}\kappa_{21}+\frac{8}{(n-1)^2}\kappa_{20}^2\kappa_{11}\right)\\&-c^3\left(\frac{1}{n^2}\kappa_{60}+\frac{12}{n(n-1)}\kappa_{40}\kappa_{20}+\frac{4(n-2)}{n(n-1)^2}\kappa_{30}^2+\frac{8}{(n-1)^2}\kappa_{20}^3\right).\end{aligned} \tag{7}$$

The expression for $\kappa_4(\psi)$ was evaluated in a similar manner. It is not recorded here because of its length.

3. The position in large samples

As the sample size increases we know, from the Central Limit Theorem, that the distribution of ψ will tend to normality. It follows that, in the limit, the probability that ψ exceeds zero can be determined by referring the limit of the ratio $\mathscr{E}(\psi)/\sigma(\psi)$ to the normal probability scale.

Using equations (4), (5) and (6)

$$\lim_{n\to\infty}\frac{\mathscr{E}(\psi)}{\sigma(\psi)}=t_{\infty,\alpha}\left\{\frac{\kappa_{20}\kappa_{02}-\kappa_{11}^2}{\kappa_{20}\kappa_{02}-\kappa_{11}^2+\kappa_{22}-2\kappa_{11}\kappa_{21}/\kappa_{20}+\kappa_{40}\kappa_{11}^2/\kappa_{20}^2}\right\}^{\frac{1}{2}}, \tag{8}$$

where $t_{\infty,\alpha}$ is the standardized normal deviate corresponding to an upper tail area of α.

It follows that, for large samples, the distribution of the regression coefficient is the same as for a bivariate normal parent distribution for all parent distributions for which

$$\kappa_{22}\kappa_{20}^2+\kappa_{40}\kappa_{11}^2=2\kappa_{21}\kappa_{11}\kappa_{20}. \tag{9}$$

Using K. Pearson's (1925) condition for linearity of regression with the addition of the condition for homoscedasticity of arrays it can be shown that equation (9) is satisfied by all distributions in which the regression of y on x is linear and the y arrays are homoscedastic.

Equation (9) is satisfied also by any distribution with zero fourth-order cumulants. This includes the third-order Gram-Charlier surfaces, considered by Cook (1951*b*). Her conclusion 'that distortion in the way described...of the normal parent population is unlikely to lead to error in tests of significance' is thus confirmed for large samples.

4. The position in finite samples

To investigate the effect of non-normality upon the distribution of the regression coefficient in finite samples it is necessary to insert numerical values for the population cumulants into the expressions for the cumulants of ψ. It will be found that $\kappa_2(\psi)$, $\kappa_3(\psi)$ and $\kappa_4(\psi)$ contain population cumulants up to the 4th, 6th and 8th orders, respectively.

While it has been found that, in univariate distributions, the main forms of departure from normality met in practice can be adequately specified in terms of the ratios $\kappa_3/\kappa_2^{\frac{3}{2}}$ and κ_4/κ_2^2, the position is very different in the bivariate case. If we go only as far as the 4th order, there are fourteen bivariate cumulants, eight of which are determined by the marginal distributions, and the remaining six by the form of association between the variables. Until much further investigation has been carried out on the values of standardized bivariate cumulants and the relations between them found in observed distributions no adequate study of the sensitivity of regression and correlation theory to departure from normality is possible.

Here, just a single example is used to illustrate the theory, taking a mathematical function to represent the population distribution and so eliminating the effect of sampling fluctuations which may make the higher cumulants of observed distributions seriously unrepresentative.

5. Application to a distribution of barometric heights

The population used is the mathematical surface fitted by E. C. Rhodes (1923) to a bivariate distribution of barometric heights recorded simultaneously at Southampton and Laudale. The form of equation was suggested by the fact that the original data are bounded, roughly, by two intersecting straight lines. The surface is defined by the equation

$$z=z_0e^{-lx-my}\left(1-\frac{x}{a}+\frac{y}{b}\right)^p\left(1+\frac{x}{a'}-\frac{y}{b'}\right)^{p'}, \tag{10}$$

the exponential term being introduced to ensure that, for x, y large, z should tend to zero.

The characteristics of this surface are: (i) increasing standard deviation in y-arrays as x increases, and vice versa; (ii) marginal distributions that tend to Type III curves as the correlation approaches unity; (iii) regression that tends to linearity as the correlation approaches unity.

In the case of the barometric heights distribution the correlation is $+0{\cdot}78$, and the regression is approximately linear.

If, following Rhodes, we define

$$s=p+1,\quad \theta=\frac{ap'}{a'p},\quad \phi=\frac{bp'}{b'p},\quad \lambda=\frac{p'+1}{p+1},$$

then the standardized cumulants of Rhodes's surface are given by

$$\kappa_{ij}=\frac{(i+j-1)!\,(\theta^j\phi^i+\lambda)}{s^{\frac{1}{2}(i+j-2)}(\theta^2+\lambda)^{\frac{1}{2}j}(\phi^2+\lambda)^{\frac{1}{2}i}}. \tag{11}$$

There is no loss of generality in standardizing the surface, so far as the comparison of the distribution of the regression coefficient with the normal theory distribution is concerned.

Rhodes's calculations gave the following values for the parameters:

$$s = 11{\cdot}88185; \quad \theta = 2{\cdot}831825; \quad \phi = 0{\cdot}680439; \quad \lambda = 1{\cdot}155645.$$

These values were used to calculate the required cumulants of 4th order and above. For the 2nd and 3rd order cumulants the figures of the observed distribution, as given by Rhodes, were used. Three of these were found to differ slightly, in the second decimal place, from the values given by (11). All the cumulants were taken to two decimal places only, and are shown in Table 1.

Table 1. *Standardized bivariate cumulants of Rhodes's surface*

κ_{11}	0·78	κ_{40}	0·26	κ_{50}	0·23	κ_{60}	0·25	κ_{80}	0·53
κ_{30}	0·41	κ_{31}	0·17	κ_{41}	0·13	κ_{51}	0·13	κ_{71}	0·25
κ_{21}	0·29	κ_{22}	0·17	κ_{32}	0·11	κ_{42}	0·10	κ_{62}	0·15
κ_{12}	0·31	κ_{13}	0·24	κ_{23}	0·15	κ_{33}	0·12	κ_{53}	0·14
κ_{03}	0·47	κ_{04}	0·39	κ_{14}	0·25	κ_{24}	0·19	κ_{44}	0·20

The moments of the criterion, ψ, were calculated for three sample sizes and two significance levels both for Rhodes's surface and for a normal surface. The expressions $\mathscr{E}(\psi)/\sigma(\psi)$, $\beta_1(\psi)$ and $\beta_2(\psi)$ were derived from the moments and are shown in Table 2.

Table 2. *The parameters of the distributions of the criterion ψ, and the types of frequency curves fitted to them*

Normal surface						
Normal theory significance level, α	0·05			0·01		
Sample size, n	11	21	61	11	21	61
$\mathscr{E}(\psi)/\sigma(\psi)$	−1·408	−1·514	−1·599	−1·738	−1·974	−2·191
$\beta_1(\psi)$	0·5372	0·1762	0·0241	0·6937	0·2658	0·0430
$\beta_2(\psi)$	3·981	3·422	3·116	4·106	3·488	3·129
Type of curve fitted	S_L	S_U	S_U	VI	S_L	S_U
Rhodes's surface						
Normal theory significance level, α	0·05			0·01		
Sample size, n	11	21	61	11	21	61
$\mathscr{E}(\psi)/\sigma(\psi)$	−1·301	−1·395	−1·465	−1·612	−1·819	−2·006
$\beta_1(\psi)$	0·9288	0·3279	0·0500	0·2004	0·4968	0·0875
$\beta_2(\psi)$	5·194	3·980	3·271	5·397	4·124	3·303
Type of curve fitted	S_U	S_U	S_U	S_U	S_U	S_U

Having found these parameters it was necessary to estimate the proportions of the distributions greater than zero. This was done using suitable frequency curves with the same parameters (remembering that $\mu_3(\psi)$ is negative). The type of curve fitted in each case is also shown in Table 2; nine were of N. L. Johnson's (1949) S_U type, two were log-normal (S_L in Johnson's notation), and one a Pearson Type VI. The method of fitting used for the S_U curves was that given by J. Draper (1952).

It should be remembered that these curves have only the first four moments fitted and will not represent the distributions of ψ precisely. It was to check the adequacy of the representation that a normal surface was considered as well as Rhodes's surface.

The proportions of the fitted curves greater than zero are shown in Table 3.

Table 3. *Comparison between normal theory and true significance levels*

Normal theory significance level, α	0·05				0·01			
Sample size, n	11	21	61	∞	11	21	61	∞
Approximation to true significance level:								
Normal surface	·052	·051	·050	·050	·009	·010	·010	·010
Rhodes's surface	·062	·062	·064	·063	·016	·014	·015	·015

For the normal populations the proportions would be 0·05 and 0·01 precisely if the fitted curves followed the exact distributions of ψ. In fact, the variation from these values is slight.

Also shown in Table 3 are the values to which these proportions tend as sample size increases, derived by assuming that ψ is normally distributed and using equation (8).

These results show that the approximate true significance levels for the finite samples for Rhodes's surface vary from the infinite sample limits by no more than the levels for the normal surface vary from their limits. It thus appears that, for sample sizes from 11 upwards, the discrepancies in these particular significance levels due to non-normality are constant, and can be calculated, using equation (8), from a knowledge of six cumulants only. It is possible that this remains true for even smaller samples but this has not yet been investigated. Whether this constancy is a general law or a freak result due to some characteristic of the particular population chosen also requires further investigation.

Whether the size of the discrepancies is to be considered as seriously affecting the validity of significance tests will depend upon the particular problem for which the tests are used. Certainly it would not be unreasonable to regard the 50 % increase in the 0·01 level as of importance, and it should be remembered that for higher significance the relative discrepancy becomes larger.

I wish to thank Dr F. N. David and Prof. E. S. Pearson for advice and help during the preparation of this paper, and to make acknowledgement to the Chief Scientist, Ministry of Supply, for permission to publish it.

REFERENCES

Cook, M. B. (1951*a*). *Biometrika*, **38**, 179.
Cook, M. B. (1951*b*). *Biometrika*, **38**, 368.
David, F. N. & Johnson, N. L. (1951). *Biometrika*, **38**, 43.
Draper, J. (1952). *Biometrika*, **39**, 290.
Johnson, N. L. (1949). *Biometrika*, **36**, 149.
Pearson, K. (1925). *Biometrika*, **17**, 268.
Rhodes, E. C. (1923). *Biometrika*, **14**, 355.

Probability distributions arising from points on a line

By P. V. KRISHNA IYER, *Defence Science Laboratory, New Delhi*
and M. N. KAPUR, *Indian Council of Agricultural Research*

For n points on a line possessing any one of two characters A or B with probabilities p and q respectively Wishart & Hirschfeld (1936) have considered the distribution of the number of links like AB and BA for adjacent points. Iyer (1951) has obtained the difference equations satisfied by the probability and moment-generating functions (p.g.f. and m.g.f.) of a number of distributions similar to those discussed by Wishart and Hirschfeld when the points can assume more than two characters. In such cases the general solutions of the m.g.f. (or p.g.f.) cannot be obtained as it may not be possible to get the roots of the characteristic equations. The writer derived the cumulants for these cases by an independent approach without making any use of the difference equations. The object of this note is to show how the cumulants can be evaluated with the aid of the characteristic equations.

We shall take for simplicity the case of three characters A, B and C occurring with probabilities $p_i (i = 1, 2, 3)$. Then the difference equation of the m.g.f. for number of adjoining links AB, BA, AC, CA, etc., reduces to

$$M(n+3) - M(n+2) - (\theta^2 - 1)\,\Sigma p_r p_s M(n+1) + p_1 p_2 p_3 (3\theta^2 - 2\theta^3 - 1)\,M(n) = 0, \qquad (1)$$

where $\theta = e^t$ and $M(n)$ represents the m.g.f. for n points.

We note that, for $t = 0$, two roots of the characteristic equation of (1) are zero and the third root is unity. This property holds good for the characteristic equation of many other distributions which arise from considerations of complex chains. If λ_1, λ_2 and λ_3 are the roots of the characteristic equation of (1)

$$M(n) = A_1 \lambda_1^n + A_2 \lambda_2^n + A_3 \lambda_3^n, \qquad (2)$$

where A_1, A_2 and A_3 are obtained from

$$\left.\begin{aligned} M(1) &= A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3, \\ M(2) &= A_1\lambda_1^2 + A_2\lambda_2^2 + A_3\lambda_3^2, \\ M(3) &= A_1\lambda_1^3 + A_2\lambda_2^3 + A_3\lambda_3^3. \end{aligned}\right\} \qquad (3)$$

Now it can be easily seen that

$$\left.\begin{aligned} M(1) &= 1, \\ M(2) &= (1 - 2\Sigma p_r p_s) + 2\Sigma p_r p_s \theta, \\ M(3) &= \{1 - 3\Sigma p_r p_s (p_r + p_s) - 6p_1 p_2 p_3\} + 2\Sigma p_r p_s (p_r + p_s)\,\theta + \{6p_1 p_2 p_3 + \Sigma p_s p_r (p_r + p_s)\}\,\theta^2. \end{aligned}\right\} \qquad (4)$$

Further
$$\begin{aligned} \log M(n) &= \log (A_1 \lambda_1^n + A_2 \lambda_2^n + A_3 \lambda_3^n) \\ &= \log A_1 + n \log \lambda_1 + \log \left\{ 1 + \frac{A_2}{A_1} \left(\frac{\lambda_2}{\lambda_1}\right)^n + \frac{A_3}{A_1} \left(\frac{\lambda_3}{\lambda_1}\right)^n \right\} \\ &= \kappa_1 t + \frac{\kappa_2}{2!} t^2 + \frac{\kappa_3}{3!} t^3 + \ldots, \end{aligned} \qquad (5)$$

where λ_1 is the root not equal to zero for $t = 0$. By differentiating (5) r times it follows that

$$\kappa_r = \left[\frac{d^r}{dt^r} \log A_1\right]_{t=0} + n \left[\frac{d^r}{dt^r} \log \lambda_1\right]_{t=0} \qquad (6)$$

because
$$\frac{d^r}{dt^r} \left[\log \left\{ 1 + \frac{A_2}{A_1} \left(\frac{\lambda_2}{\lambda_1}\right)^n + \frac{A_3}{A_1} \left(\frac{\lambda_3}{\lambda_1}\right)^n \right\} \right]_{t=0} = 0$$

for all admissible values of r.

Thus κ_r can be evaluated provided

$$\left(\frac{d^r}{dt^r} \log A_1\right)_{t=0} \quad \text{and} \quad \left(\frac{d^r}{dt^r} \log \lambda_1\right)_{t=0}$$

are known. Now it can be seen that

$$A_1 = \frac{\begin{vmatrix} M(1) & \lambda_2 & \lambda_3 \\ M(2) & \lambda_2^2 & \lambda_3^2 \\ M(3) & \lambda_2^3 & \lambda_3^3 \end{vmatrix}}{\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{vmatrix}} = \frac{u}{v}, \tag{7}$$

where $u = \lambda_1^3 + \{M(2)-1\}\lambda_1 + \{M(3)-M(2)-(\theta^2-1)\Sigma p_r p_s\}$

and $v = 3\lambda_1^3 - 2\lambda_1^2 - (\theta^2-1)\lambda_1 \Sigma p_r p_s$.

Differentiating the characteristic equation of (1) in succession and putting $t = 0$, we get

$$\left.\begin{aligned} \lambda_1' &= 2\Sigma p_r p_s, \\ \lambda_1'' &= 6p_1p_2p_3 + 4\Sigma p_r p_s - 8(\Sigma p_r p_s)^2, \\ \lambda_1''' &= 30p_1p_2p_3 + 8\Sigma p_r p_s - 48(\Sigma p_r p_s)^2 + 96(\Sigma p_r p_s)^3 - 108p_1p_2p_3\Sigma p_r p_s, \\ \lambda_1^{iv} &= 114p_1p_2p_3 - 576p_1p_2p_3(2\Sigma p_r p_s) + 720p_1p_2p_3(2\Sigma p_r p_s)^2 \\ &\quad - 432p_1^2p_2^2p_3^2 + 16\Sigma p_r p_s - 224(\Sigma p_r p_s)^2 + 144\times 8(\Sigma p_r p_s)^3 - 120\times 16(\Sigma p_r p_s)^4. \end{aligned}\right\} \tag{8}$$

Also for $t = 0$

$$\left.\begin{aligned} u' &= 6\Sigma p_r p_s, \\ u'' &= 10\Sigma p_r p_s + 18p_1p_2p_3, \\ u''' &= 78p_1p_2p_3 - 54\times 2p_1p_2p_3(\Sigma p_r p_s) + 18\Sigma p_r p_s - 12(\Sigma p_r p_s)^2 + 48(\Sigma p_r p_s)^3, \\ u^{iv} &= 270p_1p_2p_3 - 1224p_1p_2p_3(\Sigma p_r p_s) + 648\times 4p_1p_2p_3(\Sigma p_r p_s) \\ &\quad - 648p_1^2p_2^2p_3^2 + 34\Sigma p_r p_s - 96(\Sigma p_r p_s)^2 + 84\times 8(\Sigma p_r p_s)^3 - 72\times 16(\Sigma p_r p_s)^4. \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} v' &= 8\Sigma p_r p_s, \\ v'' &= 30p_1p_2p_3 + 16\Sigma p_r p_s + 8(\Sigma p_r p_s)^2, \\ v''' &= 150p_1p_2p_3 - 36p_1p_2p_3(2\Sigma p_r p_s) + 32\Sigma p_r p_s + 48(\Sigma p_r p_s)^2, \\ v^{iv} &= 570p_1p_2p_3 - 768p_1p_2p_3(\Sigma p_r p_s) + 1728p_1p_2p_3(\Sigma p_r p_s)^2 \\ &\quad - 648p_1^2p_2^2p_3^2 + 64\Sigma p_r p_s + 224(\Sigma p_r p_s)^2 - 384(\Sigma p_r p_s)^4. \end{aligned}\right\} \tag{10}$$

Using (8), (9) and (10) we get

$$\begin{aligned} \kappa_1 &= 2(n-1)\Sigma p_r p_s, \\ \kappa_2 &= n\{6p_1p_2p_3 + 4\Sigma p_r p_s - 12(\Sigma p_r p_s)^2\} + \{-12p_1p_2p_3 - 6\Sigma p_r p_s + 20(\Sigma p_r p_s)^2\}, \\ \kappa_3 &= n\{30p_1p_2p_3 - 144p_1p_2p_3\Sigma p_r p_s + 8\Sigma p_r p_s \\ &\quad - 72(\Sigma p_r p_s)^2 + 160(\Sigma p_r p_s)^3\} + \{-72p_1p_2p_3 + 360p_1p_2p_3\Sigma p_r p_s \\ &\quad - 14\Sigma p_r p_s + 144(\Sigma p_r p_s)^2 + 352(\Sigma p_r p_s)^3\}, \\ \kappa_4 &= n\{114p_1p_2p_3 - 1536p_1p_2p_3\Sigma p_r p_s + 4320p_1p_2p_3(\Sigma p_r p_s)^2 \\ &\quad - 540p_1^2p_2^2p_3^2 + 16\Sigma p_r p_s - 336(\Sigma p_r p_s)^2 + 1920(\Sigma p_r p_s)^3 - 3360(\Sigma p_r p_s)^4\} \\ &\quad + \{-300p_1p_2p_3 + 4272p_1p_2p_3\Sigma p_r p_s - 12672p_1p_2p_3(\Sigma p_r p_s)^2 \\ &\quad + 1728p_1^2p_2^2p_3^2 - 30\Sigma p_r p_s + 740(\Sigma p_r p_s)^2 - 4704(\Sigma p_r p_s)^3 \\ &\quad + 8928(\Sigma p_r p_s)^4\}. \end{aligned}$$

It may be noted that the above values after reduction tally with those given by one of the authors earlier (1951).

REFERENCES

KRISHNA IYER, P. V. (1951). Further contributions to the theory of probability distributions of points on a line. *J. Indian Soc. Agric. Statist.* **3**, 80.

WISHART, J. & HIRSCHFELD, H. O. (1936). A theorem concerning the distribution of joins between line segments. *J. Lond. Math. Soc.* **11**, 227.

The factorial moments of the distribution of joins between line segments

BY JOHN WISHART

Statistical Laboratory, University of Cambridge

The preceding paper by P. V. Krishna Iyer will recall to readers the method used by Wishart & Hirschfeld (1936) for finding the cumulants of the distribution of joins between line segments when the segments are of two kinds only, e.g. black or white. Even in this simple case, while an explicit expression could be obtained for the moment-generating function (since the characteristic equation was quadratic and could be solved directly), it was not possible to write down general expressions for the cumulants of the distribution, nor for the moments about the mean. Instead, the two additive parts of the logarithm of the moment-generating function $M_n(t)$ were expanded as far as the terms in $t^4/4!$, thus enabling the first four cumulants to be written down. Further work on generalizations of the problem, e.g. by Krishna Iyer, has likewise been limited to working out the first four cumulants.

Now this distribution is an example of a class of discontinuous distributions (of which the binomial is a very simple example) for which it is much easier to obtain general expressions for the factorial moments than for the moments about the mean or for the cumulants. Recently, in lectures, the author has been giving a new version of the old proof which produces the factorial moments, and it may be of some interest to put the results on record.

In the paper cited, the mean number of joins was first worked out and then, following this, the moment-generating function of moments about the mean was written down (p. 229, eq. (10)). If we consider, instead, moments about the origin, it is easy to see (indeed, this is implicit in the proof of the lemma on p. 233) that the recurrence formula for the moment-generating function has the simpler form

$$M'_{n+1}(t) = M'_n(t) + pq(e^{2t}-1)\,M'_{n-1}(t). \tag{1}$$

By repeating the argument in terms of factorial moments, or, in the alternative, by writing $e^t = 1+u$ in (1), we obtain the equivalent recurrence formula for the factorial moment generating function in the form

$$M_{(n+1)}(u) = M_{(n)}(u) + pqu(u+2)\,M_{(n-1)}(u). \tag{2}$$

As before, we may write down the solution as

$$M_{(n)}(u) = c_1 a^n + c_2 b^n,$$

where a and b (functions of u) are the roots of the quadratic

$$x^2 - x - pqu(u+2) = 0.$$

Thus
$$a, b = \tfrac{1}{2}[1 \pm \{1+4pqu(u+2)\}^{\frac{1}{2}}].$$

Now it is easy to see by direct computation that

$$M_{(1)}(u) = 1, \quad M_{(2)}(u) = 1+2pqu.$$

These results enable us to determine c_1 and c_2, and we then have

$$M_{(n)}(u) = \frac{a^n - b^n}{a-b} + 2pqu\,\frac{a^{n-1}-b^{n-1}}{a-b}. \tag{3}$$

We note that
$$a+b = 1, \quad a-b = \{1+4pqu(u+2)\}^{\frac{1}{2}}, \quad ab = -pqu(u+2).$$

The first term in (3) may be expanded in the form

$$\frac{a^n-b^n}{a-b} = 1 - \binom{n-2}{1}ab + \binom{n-3}{2}a^2b^2 - \binom{n-4}{3}a^3b^3 + \ldots,$$

for which the general term is obvious. By writing $n-1$ for n we have the appropriate expansion for the multiplier of $2pqu$ in the second term of (3). We then combine and simplify. Since the rth factorial moment $\mu'_{(r)}$ is the term in $u^r/r!$ in the expansion of (3), we obtain the following formulae for the factorial moments of the distribution, depending upon whether r is odd or even:

$$\left.\begin{aligned} \mu'_{(2s-1)} &= 2(2s-1)!\,(pq)^s \sum_{j=0}^{s-1} \binom{n-2s+1}{2j+1}\binom{n-s-j-1}{s-j-1}(4pq)^j, \\ \mu'_{(2s)} &= (2s)!\,(pq)^s \sum_{j=0}^{s} \binom{n-2s}{2j}\binom{n-s-j-1}{s-j}(4pq)^j. \end{aligned}\right\} \tag{4}$$

The first four of these factorial moments, in a form capable of still further simplification, are

$$\mu'_1 = 2(n-1)pq,$$

$$\mu'_{(2)} = 2pq\left\{n-2+\binom{n-2}{2}4pq\right\},$$

$$\mu'_{(3)} = 12p^2q^2\left\{(n-3)^2+\binom{n-3}{3}4pq\right\},$$

$$\mu'_{(4)} = 24p^2q^2\left\{\binom{n-3}{2}+(n-4)\binom{n-4}{2}4pq+\binom{n-4}{4}(4pq)^2\right\},$$

whence the standard formulae expressing moments (and cumulants) in terms of factorial moments would enable the known results for the former to be reached without much trouble.

REFERENCE

WISHART, J. & HIRSCHFELD, H. O. (1936). *J. Lond. Math. Soc.* **11**, 227.

Note on the use of Sherman's statistic as a test for randomness

BY D. J. BARTHOLOMEW

University College, London

1. INTRODUCTION

1·1. In investigations of the randomness of occurrence of events in time or space the following problem often arises: Given n points on a line of unit length, are they randomly distributed along this line? This is equivalent to the formal problem of whether we can regard the n points $y_1, y_2, \ldots, y_n$ as chosen at random from the rectangular population $p(y) = 1$ $(0 \leqslant y \leqslant 1)$.

Let $x_1, x_2, \ldots, x_{n+1}$ denote the successive *intervals* between the points, so that

$$\sum_{i=1}^{n+1} x_i = 1 \quad (x_1 = \min.(y_i),\ x_{n+1} = 1 - \max.(y_i)).$$

Kendall (1946) suggested the use of the criterion $w_n = \frac{1}{2}\sum_{i=1}^{n+1}\left|x_i - \frac{1}{n+1}\right|$ as the basis for a test of randomness. We note that $0 \leqslant w_n \leqslant n/(n+1)$.

1·2. It can be shown that if $t_1, t_2, \ldots, t_{n+1}$ are independent random variables with the common exponential distribution

$$p(t) = \lambda e^{-\lambda t} \quad (t \geqslant 0),$$

then the criterion $$w'_n = \frac{1}{2}\sum_{i=1}^{n+1}|t_i - \bar{t}|/\{(n+1)\bar{t}\}$$

has the same distribution at w_n. Hence the common distribution may be used as a basis for tests based on either criterion.

1·3. The moments and probability integral of w_n were obtained by Sherman (1950), who gave the following results:

$$F(x) = \Pr\{w_n \leqslant x\} = 1 + \sum_{q=0}^{n-v-1}\sum_{p=0}^{q}(-1)^{q+p+1}\binom{n}{p}\binom{n+1}{q+1}\binom{n+q-p}{n}\left(\frac{n-1}{n+q}\right)^q\left(\frac{n-q}{n+1}-x\right)^{n-p},$$

where v is the integer satisfying $\quad v/(n+1) \leqslant x \leqslant (v+1)/(n+1),$

and $$\mu'_r(w_n) = \binom{n+r}{r}^{-1}\sum_{s=0}^{g-1}\binom{n+1}{s+1}\binom{r-1}{s}\left(\frac{n-s}{n+1}\right)^{n+r},$$

where $$g = \min(r, n).$$

Expanding the moments in powers of n^{-1} we have

$$\mu_1' = \frac{1}{e}\left\{1 - \frac{1}{2n} + \frac{7}{24n^2} + O\left(\frac{1}{n^3}\right)\right\},$$

$$\mu_2 = \frac{0{\cdot}05908}{n} - \frac{0{\cdot}07145}{n^2} + O\left(\frac{1}{n^3}\right),$$

$$\beta_1 = \frac{0{\cdot}356}{n} + O\left(\frac{1}{n^2}\right), \quad \beta_2 = 3 - \frac{1{\cdot}05}{n} + O\left(\frac{1}{n^2}\right).$$

2. Approximation to the distribution of w_n

2·1. Calculation of exact percentage points is difficult. However, Pearson Type I curves fitted to the range of variation and the first two moments of w_n give results which are in close agreement with the exact values where they are available. Table 1 shows values based on Pearson Type I curves compared with exact values for $n = 3$, 5 and 10. It will be noticed that the approximation to the upper 5 % point appears to get worse as the sample size increases. As the distribution of w_n tends to normality with increasing n we may expect the fitted values eventually to converge to the true values. An inspection of the other percentage points confirms this, and suggests that we shall make little error in judging significance by using the approximation.

Table 1. *True tail probabilities corresponding to percentage points of fitted curves*

Percentage point	$n=3$	$n=5$	$n=10$
Upper 5 %	0·051	0·051	0·054
Upper 1 %	0·017	0·015	0·011
Lower 5 %	0·046	0·046	—
Lower 1 %	0·011	0·010	—

2·2. If it be assumed that the distribution of $X_n = w_n(n+1)/n$ can be approximated by the Pearson Type I distribution

$$p(X_n) = \frac{1}{B(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} X_n^{\frac{1}{2}\nu_1 - 1}(1 - X_n)^{\frac{1}{2}\nu_2 - 1} \quad (0 \leqslant X_n \leqslant 1),$$

then

$$W_n = \frac{\nu_2 X_n}{\nu_1(1 - X_n)}$$

would be distributed approximately as F with degrees of freedom ν_1, ν_2. Table 2 gives the values of ν_1 and ν_2 for $n = 3$ to $n = 20$.

Table 2

n	3	4	5	6	7	8	9	10	11
ν_1	7·70	10·59	13·50	16·40	19·29	22·19	25·09	27·95	30·83
ν_2	10·55	15·27	20·09	24·95	29·83	34·75	39·67	44·55	49·50
n	12	13	14	15	16	17	18	19	20
ν_1	33·81	36·68	40·03	42·26	45·48	48·56	51·19	53·98	57·10
ν_2	54·53	59·44	65·13	69·33	74·05	79·76	84·28	89·06	94·40

2·3. In order to find a suitable approximation when $n > 20$, the 5 % points of w_{20} were obtained by three methods. The results are shown in Table 3.

Table 3. *Approximate percentage points for* w_{20}

Percentage point	Normal	Type I (2 moments)	Gram-Charlier (3 moments)
Upper 5 %	0·4457	0·4473	0·4476
Lower 5 %	0·2722	0·2741	0·2742

It seems that the normal approximation is sufficiently accurate for most purposes if $n > 20$. If greater accuracy is required the Cornish-Fisher (1937) inversion of the Gram-Charlier series, leading to the result that

$$u' = u - \frac{0{\cdot}0995}{\sqrt{n}}(u^2 - 1),$$

where

$$u = \frac{w_n - 0{\cdot}3679(1 - \frac{1}{2}n^{-1})}{0{\cdot}2431n^{-\frac{1}{2}}(1 - 0{\cdot}605n^{-1})}$$

is approximately a unit normal variable, may be used. This transformation gives good results even for n as low as 10.

3. Application of the test

3·1. Either tail of the distribution of w_n (or W_n) may be used in testing significance, according to the alternative hypotheses envisaged. Usually it will be high values of w_n that are regarded as significant, indicating more irregularity than would be expected on the hypothesis of randomness.

3·2. The data on accidents in coal mines considered in detail by Maguire, Pearson & Wynn (1952, 1953) are in a form suitable for the application of the w_n test. In Table 1 of their 1952 paper the intervals between accidents over a period of 26,263 days are given. We obtain a value $w_{106} = 0{\cdot}4085$. The corresponding value of u is 1·82, and $u' = 1{\cdot}80$. Regarded as a unit normal variable this is significant at the upper 5 % level.

Maguire *et al.* applied Fisher's g-test, which uses the greatest interval, and Bartlett's M-test to the data without establishing significance. In later notes Barnard (1953) showed that Kolmogoroff's statistic (D_n) gave a significant result, and Maguire *et al.* (1953) also obtained significance using the ω^2-test.

From Table 3 of the 1952 paper we find, for Division 4, $w_{15} = 0{\cdot}3814$. For $n = 15$, our Table 2 gives $\nu_1 = 42{\cdot}26$, $\nu_2 = 69{\cdot}33$, whence $W_{15} = 1{\cdot}12$. Referring this value to the F-tables we see that it is not significant. The exceptionally long interval of 98 days is not sufficiently long to affect the significance of w_n.

3·3. w_n has been calculated for a number of other sets of data, including the remaining tables given by Maguire *et al.* The conclusions reached were generally in accord with those based on other tests.

In order to assess the value of the w_n-test it would be necessary to compare its power with that of other tests. Under the conditions set out in § 1·1, where the intervals are consecutive, w_n, together with other tests based on intervals, takes no account of the order in which the intervals occur. In cases where order is important more powerful tests must certainly be capable of construction. If the intervals are not ordered, w_n may be preferred among the available tests because of its ease of application.

My thanks are due to Dr F. N. David and Dr N. L. Johnson for assistance in the presentation of this work.

REFERENCES

Barnard, G. A. (1953). *Biometrika*, **40**, 212.
Cornish, E. A. & Fisher, R. A. (1937). *Rev. Inst. Int. Statist.* **5**, 307.
Kendall, M. G. (1946). *J. Roy. Statist. Soc.* **109**, 103.
Maguire, B. A., Pearson, E. S. & Wynn, A. H. A. (1952). *Biometrika*, **39**, 168.
Maguire, B. A., Pearson, E. S. & Wynn, A. H. A. (1953). *Biometrika*, **40**, 213.
Sherman, B. (1950). *Ann. Math. Statist.* **21**, 339.

Two early papers on the relation between extreme values and tensile strength

BY JULIUS LIEBLEIN

National Bureau of Standards, Washington, D.C.

Practically every writer (e.g. Epstein, 1948; Gumbel, 1954; Johnson, 1953; Epstein & Brooks, 1948) who makes a historical reference to the use of extreme values in tensile strength testing dates such application from Peirce's paper in 1926, the statistical model for which is based on Griffith's theory of flaws enunciated in 1920 (Griffith, 1920, p. 179). In the interests of historical accuracy, it therefore seems important to report any material that comes to light which would significantly affect the widely held belief concerning priority. The purpose of this note is to bring to light two apparently forgotten articles written by W. S. Chaplin, Professor of Civil Engineering, University of Tokyo, that appeared in American engineering journals over 70 years ago and recently came to the attention of the writer.* The first (Chaplin, 1880) was published in *Van Nostrand's Engineering Magazine* for December 1880 and was entitled 'The relation between the tensile strengths of long and short bars'. The second (Chaplin, 1882), appearing in *Proceedings of the Engineers' Club* for 1882, bore the title 'On the relative tensile strengths of long and short bars', and was essentially an elaboration of the first. It will be sufficient to confine attention to the first of the two articles.

This article essentially gave the basis of the application of extreme values to tensile strength. The author began by mentioning the applicability of the (Gaussian) 'law of errors' to the variation in strengths of *individual* specimens or pieces of a given size, and characterized their dispersion by their 'probable variation', i.e. probable error, where nowadays we would use standard deviation. He then showed that p^n is the chance that each of n pieces of the same size would exceed a certain tensile strength if p is the probability that each piece exceeds it. He next considered the key question that was the purpose of the article (p. 442):

> Suppose that many pieces of cross section c, and length one inch have been tested for tensile strength with an average result S_0, and a probable variation in one piece of P_0; what will be the probable average† strength, S_n, of pieces of the same cross section and a length of n inches?
>
> Knowing the probable variation in a piece one inch long, we are able to construct the curve showing the probability of any and all variations in a piece of this length. From this curve we can obtain the probability that the piece one inch long will break between any limits of variation. The probability that an inch-piece will break above a negative variation $-x$ is $0{\cdot}5+A_x$, in which A_x represents the probability that the piece will break between 0 and $-x$. In a piece n inches long there are n pieces one inch long; the probability that any one of these will break above $-x$ being $0{\cdot}5+A_x$, the probability that all of them will break above this limit, or that the *strength of the whole piece will be at least* S_0-x, will be
>
> $$(0{\cdot}5+A_x)^n.$$
>
> As S_x‡ is an average, it is as probable that a piece n inches long will break above it as below it; hence the probability that a piece n inches long will break above it is 0·5. We have then
>
> $$(0{\cdot}5+A_x)^n = 0{\cdot}5,$$
>
> in which A_x is the unknown quantity. We easily obtain
>
> $$A_x = \sqrt[n]{0{\cdot}5}-0{\cdot}5 \quad \text{or} \quad 2A_x = 2(\sqrt[n]{0{\cdot}5}-0{\cdot}5). \qquad \text{(Italics inserted.)}$$

In (not very different) modern symbols and terminology this stated essentially that if $\Phi(x)$ is the (Gaussian) probability that the tensile strength of a piece of given length is greater than x, then the probability that a piece n times as long has strength greater than y is $[\Phi(y)]^n$. Putting $[\Phi(y)]^n = \frac{1}{2}$ and using the appropriate normal tables yielded the median value of the strength y of a piece n inches long (of similar cross-section). The author then converted the relation $[\Phi(y)]^n = \frac{1}{2}$ into a useful table and chart, and explained that if $\frac{1}{2}$ were replaced by $\frac{1}{4}$ in this relation the corresponding value of y would give the 'probable variation' (semi-interquartile range) for strengths of the long specimen. This theory was compared with the results of tests on five specimens of annealed Japanese copper wire of lengths 1, 4, 8, 12, 16 in., and agreement of theory with experiment was found to be very satisfactory.

* The writer is indebted to Dr Churchill Eisenhart for calling his attention to an old engineering text (Slocum & Hancock, 1906) in which these articles were cited.

† The meaning intended here is apparently 'median strength'. J.L.

‡ This should apparently be S_n. J.L.

The italicized portion in the above quotation is the essence of the 'weakest-link' concept—the idea that a chain is no stronger than its weakest link. Moreover, the function $[\Phi(y)]^n$ is recognized as essentially the (exact) c.d.f. of the distribution of largest values in samples of size n from the normal distribution. Thus, while the idea of 'weakest link' is probably very old, Prof. Chaplin's work appears to be the first to apply the idea statistically, through the use of extreme values, to the study of tensile strengths.

REFERENCES

CHAPLIN, W. S. (1880). The relation between the tensile strengths of long and short bars. *Van Nostrand's Engineering Magazine*, **23** (New York), 441.

CHAPLIN, W. S. (1882). On the relative tensile strengths of long and short bars. *Proc. Engineers' Club, Philadelphia*, **3**, 15.

EPSTEIN, B. (1948). Application of the theory of extreme values in fracture problems. *J. Amer. Statist. Ass.* **43**, 403.

EPSTEIN, B. & BROOKS, H. (1948). The theory of extreme values and its implications in the study of the dielectric strength of paper capacitors. *J. Appl. Phys.* **19**, 544.

GRIFFITH, A. A. (1920). The phenomena of rupture and flow in solids. *Phil. Trans.* A, **221**, 163.

GUMBEL, E. J. (1954). *Statistical theory of extreme values and some practical applications.* National Bureau of Standards Applied Mathematics Series, **33**, 6. U.S. Government Printing Office, Washington, D.C.

JOHNSON, A. I. (1953). Strength, safety and economical dimensions of structures. *Bulletin of the Division of Building Statics and Structural Engineering at the Royal Institute of Technology*, Stockholm, no. 12, p. 58.

PEIRCE, F. T. (1926). Tensile tests for cotton yarns. V. 'The weakest link'—theorems on the strength of long and of composite specimens. *J. Text. Inst., Trans.* **17**, 355.

SLOCUM, C. E. & HANCOCK, E. L. (1906). *Text-Book on the Strength of Materials*, p. 15. Boston: Ginn and Co.

Two problems in sets of measurements

BY M. G. KENDALL

Visiting Professor, Institute of Statistics, North Carolina State College and Division of Research Techniques, London School of Economics

1. Dr W. J. Youden (1953) has recently considered some questions arising out of the editing and internal consistency of a set of measurements, and in personal discussion raised two problems the solutions of which are presented herewith. The first problem (the Angel Problem) is as follows:

n members are drawn at random from a normal population with unit variance. A benevolent angel tells us which is nearest to the true mean, and the others are rejected. What is the variance of the retained member?

This is the problem as posed by Dr Youden, but the practical situation which suggested it is not exactly angelic. A number of laboratory assistants are given a standard experiment to perform. They replicate it and, knowing what the true result ought to be, each submits only his best result. What effect does this have on estimates of experimental error?

2. For $n = 1$ the variance is, trivially, unity. For $n = 2$ the solution was obtained by one of Dr Youden's colleagues at the National Bureau of Standards at Washington, Mr E. P. King. For $n = 3$ the solution was obtained by Dr H. Fairfield Smith at North Carolina State College, where I carried out this work. Below I give exact results for $n = 4$ and 5, an adequate approximation for higher values of n and an asymptotic result for large n.

3. From considerations of symmetry we see that the mean of the variate is the parent mean, which we may take to be zero. The problem then reduces to finding $E(x^2)$ in the distribution of the smallest of n members chosen from half the normal distribution, namely,

$$dF = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2} dx \quad (0 \leqslant x \leqslant \infty). \tag{1}$$

If

$$F = \int_0^x dF, \tag{2}$$

the distribution of the smallest member is

$$dG = n(1-F)^{n-1}\,dF, \tag{3}$$

and the required variance is then v_n, say, given by

$$v_n = \int_0^\infty nx^2(1-F)^{n-1}\,dF. \tag{4}$$

The only problem is now to evaluate these integrals. They closely resemble those studied by Hojo (1931), and certain relations exist between the two. I have not, however, succeeded in adapting Hojo's integrals for the present purposes, except that when a result is known for odd n they can be used to find the result for the next highest even n.* Unfortunately, they do not take us up from an even n to an odd n. We have

$$v_n = n\left(\frac{2}{\pi}\right)^{\frac{1}{2}n}\int_0^\infty x^2 e^{-\frac{1}{2}x^2}\left(\int_0^\infty e^{-\frac{1}{2}t^2}dt\right)^{n-1}dx.$$

By a partial integration with $xe^{-\frac{1}{2}x^2}$ for one part this reduces to

$$v_n = 1-n(n-1)\left(\frac{2}{\pi}\right)^{\frac{1}{2}n}\int_0^\infty x e^{-x^2}\left(\int_x^\infty e^{-\frac{1}{2}t^2}dt\right)^{n-2}dx, \tag{5}$$

and, by a further partial integration, to

$$v_n = 1-\frac{n(n-1)}{\pi}+\tfrac{1}{2}n(n-1)(n-2)\left(\frac{2}{\pi}\right)^{\frac{1}{2}n}\int_0^\infty e^{-\frac{1}{2}x^2}\left(\int_x^\infty e^{-\frac{1}{2}t^2}dt\right)^{n-3}dx. \tag{6}$$

For $n = 2$ we have from (5) immediately

$$v_2 = 1-\frac{2}{\pi}, \tag{7}$$

and for $n = 3$, from (6),

$$v_3 = 1-\frac{2}{\pi}(3-\sqrt{3}). \tag{8}$$

4. For higher n I used a reduction formula obtained as follows:

Let
$$P(\alpha,n,p) = \int_0^\infty e^{-\frac{1}{2}\alpha x^2}\left(\int_{px}^\infty e^{-\frac{1}{2}t^2}dt\right)^{n-3}dx. \tag{9}$$

Then
$$\frac{\partial P}{\partial p} = -(n-3)\int_0^\infty x e^{-\frac{1}{2}(\alpha+p^2)x^2}\left(\int_{px}^\infty e^{-\frac{1}{2}t^2}dt\right)^{n-4}dx, \tag{10}$$

which, by a further partial integration, gives after a little reduction

$$\frac{\partial P}{\partial p}(\alpha,n,p) = -\frac{(n-3)}{\alpha+p^2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}(n-4)}+\frac{p(n-3)(n-4)}{\alpha+p^2}P(\alpha+2p^2,n-2,p). \tag{11}$$

5. For $n = 4$ we use (10) to obtain

$$\frac{\partial P}{\partial p}(\alpha,4,p) = \frac{1}{\alpha+p^2}.$$

When $p = \infty$, $P = 0$, and we find, putting $\alpha = 3$, $p = 1$,

$$P(3,4,1) = \frac{1}{\sqrt{3}}\tan^{-1}\frac{p}{\sqrt{3}}\Big]_\infty^{p=1} = \frac{\pi}{3\sqrt{3}}, \tag{12}$$

and hence, from (6),

$$v_4 = 1-\frac{12}{\pi}+\frac{16}{\pi\sqrt{3}}.$$

We have at once
$$P(\alpha,3,p) = \int_0^\infty e^{-\frac{1}{2}\alpha x^2}dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\alpha}}.$$

Hence, from (11),
$$\frac{\partial}{\partial p}P(\alpha,5,p) = 2\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{-\frac{1}{\alpha+p^2}+\frac{p}{(\alpha+p^2)(\alpha+2p^2)^{\frac{1}{2}}}\right\}.$$

* It may be noted that Hojo's values for the integral he calls S (1931, p. 326) are in error, apparently by the omission of a factor 2π.

Integrating from ∞ to 1 and putting $\alpha = 3$ we find

$$P(3,5,1) = 2\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left[-\frac{1}{\sqrt{\alpha}}\tan^{-1}\frac{p}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{(\alpha+2p^2)}}{\sqrt{2}}\right]_{\infty}^{1}$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{\tan^{-1}\sqrt{\frac{5}{3}}-\frac{\pi}{6}\right\},$$

and hence

$$v_5 = 1-\frac{20}{\pi}+\frac{240}{\pi^2\sqrt{3}}\left\{\tan^{-1}\sqrt{\frac{5}{3}}-\frac{\pi}{6}\right\}. \tag{13}$$

6. For higher n the integrals become more complicated, and I have not succeeded in reducing them to elementary forms. Simple quadratures or expansions would, no doubt, evaluate them if the labour were thought worth while.

7. An asymptotic result may be obtained as follows: In the original integral defining v_n, as n becomes large the value

$$\left(\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_x^{\infty} e^{-\frac{1}{2}t^2}\,dt\right)^{n-1}$$

tends to

$$\left(1-x\sqrt{\frac{2}{\pi}}\right)^{n-1} \sim \exp\left\{-(n-1)x\sqrt{\frac{2}{\pi}}\right\}.$$

Thus

$$v_n \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} n\int_0^{\infty} x^2 \exp\left\{-(n-1)x\sqrt{\frac{2}{\pi}}\right\}dx \sim \pi/n^2. \tag{14}$$

A better formula, derived from the expansion in the following paragraph, is

$$v_n \sim \frac{\pi}{(n-1)(n+4)}. \tag{15}$$

8. The gap between the exact results and the asymptotic value may be bridged by an expansion due to Karl Pearson (1931, p. 361). In my notation his result, together with an extra term, is

$$x^2 = \frac{\pi}{2}F^2+\frac{1}{3}\left(\frac{\pi}{2}\right)^2F^4+\frac{13}{93}\left(\frac{\pi}{2}\right)^3F^6+\frac{22}{315}\left(\frac{\pi}{2}\right)^4F^8+\ldots. \tag{16}$$

On substitution in (4) we find

$$v_n = \frac{\pi}{2}\left\{\frac{2}{(n+1)(n+2)}+\frac{8}{(n+1)(n+2)(n+3)(n+4)}\left(\frac{\pi}{2}\right)+\frac{104}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\left(\frac{\pi}{2}\right)^2\right.$$

$$\left.+\frac{2816}{(n+1)(n+2)\ldots(n+8)}\left(\frac{\pi}{2}\right)^3\right\}. \tag{17}$$

For $n = 4$ this gives 0·1193 against the true value of 0·1208 and for $n = 5$ it gives 0·08261 against 0·08308. It would seem to provide, therefore, a satisfactory approximation for higher n and it agrees with the asymptotic formula. The approximate values below may be in error of 2 in the last place for $n = 6$ and 1 in the last place for $n = 7$; the remainder should be accurate to the places given.

9. Table 1 gives the values of v_n together with some sampling results obtained by Dr Youden. The agreement with experiment is reasonably good.

Dr Youden's samples are not all independent, in the sense that the material for sets of 10 was constructed from the 200 sets of 5 by pairing them and choosing the smaller of the two smallest values in the sets of 5. For the purposes of a general comparison, however, this is not important. The variance of the mean of a sample of n being $1/n$, the effect of choosing the closest member is to reduce the variance by a factor approximately equal to $\pi/(n+3)$.

10. Dr Youden's second problem (the Demon Problem) is as follows: given a (small) sample of n values from a normal population, what is the probability that their mean lies between the nth and the $(n-1)$th in order of magnitude?

The problem is trivial for $n = 2$ and 3, and not of much interest for $n > 10$. I shall give a general method of solution and obtain the probabilities explicitly for $n = 4$ to 10.

Table 1. *Values of* v_n

n	Exact	Approximate (eqn. (17))	Asymptotic (eqn. (15))	Sample value
2	0·3634	—	—	—
3	0·1928	—	—	0·189 (200 samples)
4	0·1208	0·1193	—	0·122 (200 samples)
5	0·08308	0·08261	0·08727	0·0927 (200 samples)
6	—	0·06065	0·06283	—
7	—	0·04647	0·04760	—
8	—	0·03676	0·03740	0·0374 (100 samples)
9	—	0·02982	0·03021	—
10	—	0·02469	0·02493	0·0303 (100 samples)
20	—	0·006880	0·006889	0·00768 (90 samples)

11. The deviations of order statistics from their mean are independent of the mean itself in normal samples. If

$$X = x_j - \bar{x}, \tag{18}$$

we then have that the c.f. of $x_j - \bar{x}$ multiplied by the c.f. of $\bar{x}$ is the c.f. of x_j and hence for the cumulants

$$\kappa_r(X) = \kappa_r(x_j) \quad (r = 1, 3, \ldots), \tag{19}$$

$$\kappa_2(X) = \kappa_2(x_j) - \frac{1}{n}. \tag{20}$$

This result is due to McKay (1935) who arrived at it by a longer route.

We can also use McKay's method to obtain an explicit form for the frequency function, but as I found it rather troublesome to handle and have adopted a different approach I record the result without giving the details of the derivation. If

$$A(x) = \frac{\alpha_x^{j-1}(1-\alpha_x)^{n-j}}{B(j, n-j+1)}, \tag{21}$$

where, as usual,

$$\alpha_x = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt, \tag{22}$$

the frequency function of X is

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \left(\frac{n}{n-1}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{n}{n-1}X^2\right) \exp\left(-\frac{1}{2}\frac{n-1}{n^2}D^2\right) A\left(\frac{nX}{n-1}\right), \tag{23}$$

where $D = \dfrac{d}{dx}$.

12. It is also interesting to note that McKay's result expressed in (19) and (20) has a converse. In fact, if these expressions are true for any one n and for all the order statistics x_j, the distribution of $\bar{x}$ is independent of all the deviations $x_j - \bar{x}$ and hence of the variance. Thus equations (19) and (20) are true only for normal variation; for the independence of mean and variance is a sufficient condition for parent normality.

13. I proceed by finding the moments of x_{n-1}, hence those of $x_{n-1} - \bar{x}\,(= X)$, hence the cumulants of X, and hence Prob. $(X > 0)$ from the Edgeworth form of the Gram-Charlier expansion.

The moments of the extreme member of a normal sample have recently been well tabulated by Ruben (1954). The distribution of the next-to-the-largest-value is

$$dG = n(n-1)(1-\alpha)\alpha^{n-2} d\alpha, \tag{24}$$

and hence the moments about zero are given by

$$\begin{aligned} \mu_r' &= n(n-1)\, E\{x^r(1-\alpha)\alpha^{n-2}\} \\ &= nM_r(n-1) - (n-1)M_r(n), \end{aligned} \tag{25}$$

where $M_r(n)$ is the rth moment of the extreme value is samples of n as tabulated by Ruben. The use of (26) then gives us the moments we require; and for the standardized cumulants of X I find, as far as κ_6 (Table 2).

Table 2

Values of n	κ_1	κ_3	κ_4	κ_5	κ_6
4	0·893,675	0·386,480	0·467,153	0·158,270	−0·421,023
5	1·482,339	·501,767	·457,020	·247,879	− ·212,405
6	1·909,858	·537,666	·450,896	·296,310	− ·074,065
7	2·244,373	·547,896	·445,470	·325,307	·022,561
8	2·519,644	·548,365	·440,794	·344,593	·093,139
9	2·754,160	·546,647	·436,932	·358,530	·147,552
10	2·958,969	·543,163	·433,825	·369,352	− ·190,471

The distribution function of X is

$$F(X) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}t^2\,dt\right) - \tfrac{1}{6}\kappa_3 D^2 + \tfrac{1}{24}\kappa_4 D^3 - \tfrac{1}{120}\kappa_5 D^4 + (\tfrac{1}{720}\kappa_6 + \tfrac{1}{72}\kappa_3^2)\, D^5 + \ldots, \qquad (26)$$

where D^j is given by

$$\left[\left(\frac{d}{dt}\right)^j \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}t^2}\right]_{t=x}. \qquad (27)$$

For example, with $n = 4$, I find

$$\text{Prob.}\,(x_3 > \bar{x}) = 0{\cdot}1743.$$

Thus the probability that the sample mean falls between the third and fourth observations (or between the first and second) is approximately 0·1743, and the probability that it falls between the second and third is 0·6514. Dr Youden has kindly supplied the results of a sample of 7000 which give proportions 0·1744, 0·6546, 0·1710 falling between the first and second, second and third, third and fourth observations respectively. These are well within sampling limits, yielding a χ^2 of 0·54 with 2 degrees of freedom.

14. Asymptotic results are not of much interest, but it may be worth recording that from (23) we can ascertain that the probability of the sample mean falling between the greatest-but-one value is, in the limit, $n/2^n$. This is in accordance with the known result for the probability that the *true* mean lies between those two sample members.

15. The distribution of X does not tend to normality for large n, and the Edgeworth series may therefore become less satisfactory for larger n. The following are the probabilities calculated from (27) on the basis of the terms there shown:

n	4	5	6	7	8	9	10
P	0·174	0·078	0·040	0·024	0·015	0·010	0·007

REFERENCES

Hojo, T. (1931). Distribution of the median, quartiles and interquartile distance in samples from a normal population. *Biometrika*, **23**, 315.

McKay, A. T. (1935). The distribution of the difference between the extreme observation and the sample mean in samples of n from a normal universe. *Biometrika*, **27**, 466.

Pearson, K. (1931). Appendix to a paper by Professor Tokishige Hojo. *Biometrika*, **23**, 361.

Ruben, H. (1954). On the moments of order statistics in samples from normal populations. *Biometrika*, **44**, 200.

Youden, W. J. (1953). Sets of three measurements. *Sci. Mon.* **57**, 143.

REVIEWS

Exercises in Theoretical Statistics. By M. G. KENDALL. London: Charles Griffin and Co. Ltd. 1953. Pp. vii + 179. 20*s*.

Statistical Tables and Problems. 3rd edition. By A. E. WAUGH. London: McGraw-Hill Publishing Company Ltd. 1953. Pp. xiv + 242. 25*s*. 6*d*.

In mathematics, as possibly in no other subject, must the student learn by repetition. A theorem containing a basic piece of information is proved; it is then vital to the student thoroughly to digest this information by practising its application in as many ways as he can find. 'L'Appetit vient en mangeant', and understanding grows only with seemingly unnecessary repetition. Mathematical statistics theorems are newcomers to the corpus of mathematical knowledge, and exercises and examples, which have been thought out through the centuries in other branches, are difficult to come by. Until 1946 the only source for the student was the examination papers set in the various degree examinations of the University of London. Since then we have besides this source the examination papers of the Oxford, the Cambridge and the Royal Statistical Society diplomas (to mention a few newly sprung examinations.)

Prof. Kendall has done students a service by collecting from published examination papers such questions as seem to him useful and by adding some of his own invention. There are 100 questions each on distribution theory and on sampling, 75 questions on statistical relationship, 75 on estimation and inference and 50 on time series. Hints for solution are given at the end of the book. The questions are uneven in that some are quite hard while others are easy, but taken as a whole the book will prove useful to the teacher and helpful to the student.

Prof. Waugh's book of tables and problems is intended principally for the statistics student as opposed to the would-be mathematical statistician whom Prof. Kendall must envisage. Prof. Waugh gives exercises on graphical presentation, significant figures, frequency distributions, measures of central tendency, measures of dispersion, probability and the normal curve, measures of reliability, and on correlation. The scope and standard of the exercises are such as to render them suitable for a class of students just beginning to learn statistics. No theoretical derivation is involved and the use of a calculating machine is desirable to lighten arithmetical labour. Probably the book will be found useful to teachers beginning a statistics course *ab initio* and to non-university students who are trying to learn by themselves. Answers do not appear to be given.

The tables, which are presumably included in order that the student may carry out the exercises without reference to outside publications, are the usual statistical collection with short auxiliary tables like those given in Barlow plus tables of the Perpetual Calendar and the Julian Day and a list of the Greek alphabet.

F. N. DAVID

Sampling Techniques. By W. G. COCHRAN. New York: John Wiley and Sons Inc.; London: Chapman and Hall. 1953. Pp. ix + 330. 52*s*.

Prof. Cochran's book deals entirely with that aspect of statistics which concerns itself with what are sometimes called sampling surveys, a somewhat narrower field than the title of the book might suggest. It is clearly written, with lucidity and logic in the development of the ideas put forward, and will take its place with the other *vade-mecum* of the would-be sample surveyor, Deming on *Some Theory of Sampling* and Yates on *Sampling Methods for Censuses and Surveys*.

The book begins with a short discourse on the advantages to be gained by using valid sampling techniques. We then start immediately on sampling from a finite population with the standard errors of various estimates worked out. There is some discussion on the validity of the normal approximation for the sample mean even when the sampling fraction is not small. Rarer still in such statistical literature as this, there is also a recognition that non-normality in the population may make a difference to the distribution of the variance, although we are left a little uncertain as to what use we should make of this information except to be cautious.

Chapters follow on sampling for proportions and percentages, the estimation of sample size, stratified random and systematic sampling, ratio and regression estimates, double sampling and sources of error in surveys. The whole field appears to be well covered; authors from whom the various techniques originated are acknowledged in the text and a full reference given at the end of each chapter. This

enables the student wishing to find out things for himself to go to the first beginnings. Also at the end of each chapter are exercises with solutions at the end of the book.

The would-be worker in sample surveys will find his subject presented with freshness and imagination in Prof. Cochran's book. This is perhaps somewhat surprising in view of the well-trodden track to the well, created by both theoretical and practical workers in the past ten years. However, in this book everything that could and should be said has been written down. The applied statistician will find the discussion and results useful; the theoretical statistician will find out where to go in the literature in order to develop his own problems a stage further. Perhaps the one flaw is in the algebraic formulation of the sampling problem which is not elegant.

F. N. DAVID

Statistical Tables. Edited by the Research Association of Statistical Sciences, Kyusyu University, Japan. Tokyo: Kawade Publishing Co. 1952. Pp. v + 264 (text) + 214 (tables).

This is a handsomely bound book of tables with an extensive Introduction, which, judging by the single page in English entitled 'Acknowledgement', and notes on the actual tables, is a compilation of a large number of statistical tables reproduced with permission, from many sources, together with a dozen or so original tables and diagrams which were prepared in Japan. The aim is obviously to have a comprehensive set of all tables likely to be frequently consulted by users of statistical methods. One-third of the tables and charts are taken from *Tables for Statisticians and Biometricians*, from *Biometrika*, or from Miss David's table of the correlation coefficient; two are from *Statistical Tables* by Fisher and Yates, while the remainder of the reproduced tables come from miscellaneous sources, of which the text-books of Kendall, Fry, Rietz and Goulden may be singled out for mention. There is a slight bias among the more specialized tables on the industrial applications side. The tables are clearly printed (or photographed), and will undoubtedly prove of great service in their country of origin.

The non-Japanese tables comprise, in brief, the following: factorials and log-factorials; binomial coefficients; Poisson terms and their sums; normal deviate given area, ordinate and area given deviate, ordinate given area; deviate, ordinate and sundry ratios given area; area, ordinate and differentials given deviate; percentage points of β ratios; test for outlying observations; for the maximum of a set of variance estimates; charts for the correlation coefficient; nomogram for $r = \tanh z$; incomplete normal moment functions; χ^2 percentage points, etc.; goodness of fit (Kolmogoroff) test; t, incomplete Beta and F percentage points; P_λ of sample; test of randomness in run theory; quality control test; tests on the distribution of range; ratios of factorial series and sundry constants. What appear to be the purely Japanese contributions comprise: table for likelihood solution of Type III distribution, sampling inspection tables, goodness of fit (ω^2) test, power function chart and tables, chart (with tables) of binomial probability paper, tables of random numbers, random normal and random Beta deviates, orthogonal polynomials, Anscombe's inverse sine transformation, binomial probabilities and sums, and the simple binomial inverse sine transformation.

The most extensive of the Japanese tables are 40–42, 43 and 46. The first set consist of 4 pages of random numbers, followed by 15 pages of what are called 'random samples from a normal population', presumably what are usually called random normal deviates, followed by 8 pages of 'random samples from a Beta (i.e. Type I) distribution'. Table 43 is an extensive one of orthogonal polynomials. These contain the same figures as the well-known tables of Fisher and Yates, but stop at $n = 51$. In the early stages, however, they are more extensive; up to $n = 15$ the polynomials are tabulated to degree $n-1$, while they go as far as the 9th degree up to $n = 21$ and the 7th degree up to $n = 31$. Tables 46·1 and 46·2 contain the individual terms, and their cumulative sums, of the binomial distribution, up to $n = 50$. They are similar to other such tables which have been published, except for a curious omission of the central term for even n; such terms can, however, be deduced from the cumulative sums.

The long introduction is in Japanese. After some 30 pages on basic mathematical statistical theory the plan is to devote separate sections to individual tables, or groups of tables. The discussion appears to be exceedingly comprehensive, and is well documented with references to the literature of the subject.

J. WISHART

Tables for the Design of Factorial Experiments. By T. KITAGAWA and M. MITOME. Tokyo: Baifukan Co. Ltd. Rutland, Vermont, U.S.A.: Charles E. Tuttle Co. 1953. Pp. x + 292 (text) + 253 (tables). $15.

This book contains 124 pages of theoretical introduction, 168 pages of description of tables, and eight sets of tables covering some 253 pages altogether. The introduction and description are in Japanese and therefore will not be read by many in this country. However, there is an exhaustive set of references, including many sources in English, and an index in Japanese and English. The tables themselves are annotated entirely in English and form a remarkably thorough collection of experimental designs. They are grouped as follows:

(i) *Latin squares and cubes.* This includes all squares up to 5×5, and all transformation sets up to 7×7, with examples of higher order squares. There are, further, sets of orthogonal Latin squares, orthogonal partitions in Latin squares, and the orthogonal Latin cubes up to $7 \times 7 \times 7$.

(ii) *Factorial designs.* A wide range of complete and balanced confounded designs.

(iii) *Fractional replications in factorial designs.* Fractional replication of 2^n in $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{16}$ replication, and $\frac{1}{3}$ replication of 3^n designs.

(iv) *Factorial designs with split-plot confounding.*

(v) *Factorial designs confounded in quasi-Latin squares.* These include the various plaid and half-plaid designs.

(vi) *Lattice designs.*

(vii) *Balanced incomplete block designs.*

(viii) *Youden's squares.*

The tables are not noticeably more complete than those in *Experimental Design* (Cochran & Cox)—except in the remarkably thorough documentation of Latin Squares. It is, of course, convenient to have all the designs tabled together.

It is a pity that the introduction is in Japanese, for it appears to be an unusually painstaking compilation of theory and practice of the analysis of variance, including more recent developments, such as Pearson & Hartley's power function charts, Hartley & David's work on the use of range, and calculations on the results of applying various rules for pooling sums of squares.

N. L. JOHNSON

(*All Rights reserved*)

BIOMETRIKA. Vol. 41, Parts 3 and 4

CONTENTS

PAGE

The design of an experiment in which certain treatment arrangements are inadmissible. By D. R. Cox 287

The estimation of location and scale parameters from grouped data. By J. M. Hammersley and K. W. Morton 296

Transformations of the binomial, negative binomial, Poisson and χ^2 distributions. By Gunnar Blom 302

A quickly convergent expansion for cumulative hypergeometric probabilities, direct and inverse. By M. E. Wise 317

The quotient of a rectangular or triangular and a general variate. By S. R. Broadbent . . 330

An estimation problem in quantitative assay. By J. Aitchison and J. A. C. Brown . . . 338

The use of the Hankel transform in statistics. II. Methods of computation. By R. D. Lord . . 344

A reduction formula for normal multivariate integrals. By R. L. Plackett 351

On the comparison of two means: further discussion of iterative methods for calculating tables. By W. H. Trickett and B. L. Welch 361

Some further results in the theory of pedestrians and road traffic. By Alan J. Mayne . . . 375

Bias in the estimation of autocorrelations. By F. H. C. Marriott and J. A. Pope 390

Note on bias in the estimation of autocorrelation. By M. G. Kendall 403

Tests of hypotheses in the linear autoregressive model. I. Null hypothesis distributions in the Yule scheme. By G. M. Jenkins 405

Some problems of optimum sampling. By Paul N. Somerville 420

A test for Markoff chains. By Paul G. Hoel 430

On stationary processes in the plane. By P. Whittle 434

The goodness of fit of frequency distributions obtained from stochastic processes. By V. N. Patankar 450

The distribution of range in certain non-normal populations. By H. A. David 463

The mean and coefficient of variation of range in small samples from non-normal populations. By D. R. Cox 469

The distribution of the ratio, in a single normal sample, of range to standard deviation. By H. A. David, H. O. Hartley and E. S. Pearson 482

Sampling with control variables. By E. C. Fieller and H. O. Hartley 494

Rank analysis of incomplete block designs. II. Additional tables for the method of paired comparisons. By Ralph Allan Bradley 502

Miscellanea

Approximate formulae for the percentage points and the probability integral of the non-central χ^2 distribution. By S. H. Abdel-Aty 538

Moments of the mean deviation. By A. R. Kamat 541

On the relation between estimating efficiency and the power of tests. By R. M. Sundrum . . 542

The interpretation of negative components of variance. By J. A. Nelder 544

The distribution of the regression coefficient in samples from a non-normal population. By I. D. Hill 548

Probability distributions arising from points on a line. By P. V. Krishna Iyer and M. N. Kapur 553

The factorial moments of the distribution of joins between line segments. By John Wishart . 555

Note on the use of Sherman's statistic as a test for randomness. By D. J. Bartholomew . . 556

Two early papers on the relation between extreme values and tensile strength. By Julius Lieblein 559

Two problems in sets of measurements. By M. G. Kendall 560

Reviews

M. G. Kendall's 'Exercises in Theoretical Statistics 565

A. E. Waugh's 'Statistical Tables and Problems' 565

W. G. Cochran's 'Sampling Techniques' 565

Research Association of Statistical Sciences, Kyusyu University, Japan (edd.) 'Statistical Tables' 566

T. Kitagawa and M. Mitome's 'Tables for the Design of Factorial Experiments' 567

Corrigenda 568

Originally printed in Great Britain at the University Press, Cambridge (Brooke Crutchley, University Printer)
Reprinted by Photo-Lithography by Warren and Son Ltd., Winchester.

PAGE
287
206
302
317
330
338
344
351
361
375
390
403
405
420
430
434
450
463
469
482
494
502
538
541
542
544
548
553
555
556
559
560
565
565
565
566
567
568